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# Model Predictive Control Implementation for Modified Quadruple Tank System

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# Model Predictive Control Implementation for Modified Quadruple Tank System

**Sazuan Nazrah Mohd. Azam**

Kgs. Lyngby, 2018

**DTU Compute**
**Department of Applied Mathematics and Computer Science**
**Technical University of Denmark**

Matematiktorvet
Building 303B
2800 Kongens Lyngby, Denmark
Phone +45 4525 3031
compute@compute.dtu.dk
www.compute.dtu.dk

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# Preface

This thesis was submitted to the Technical University of Denmark (DTU) in partial fulfillment of the requirements for the degree of Doctor of Philosophy (PhD) in Control System Engineering. **Part I of this thesis is an extraction of published and submitted papers, compiled in Part II.** The work presented in this thesis was carried out from November 2013 to August 2018, at the Scientific Computing section at the Department of Applied Mathematics and Computer Science (DTU Compute). This project has been supervised by Associate Professor John Bagterp Jørgensen. The project was funded by Universiti Teknikal Malaysia Melaka, UTeM and Ministry of Higher Education of Malaysia.

# Abstract


The implementation of a Model Predictive Controller (MPC) for a Modified Quadruple Tank (MQT) system is addressed in this thesis. The purpose of this study is to demonstrate the application of MPC to a multi-input-multi-output (MIMO) dynamical system that has complicated variables interactions. The basic understanding of the system and the controller is presented.

First, we investigate the behaviour of the dynamical system and determined the parameters and defined the variables that govern the MQT system, and then we derived all the formulations related to acquiring the mathematical model from the basic physical process. Then, the dynamic model of the system that has deterministic and stochastic components described as a linear discrete-time state space model derived is employed for the purpose of the next study. We specify the model in a form that is appropriate for computational operation and analysis of MPC by introducing a discrete impulse coefficient or known as Markov parameters.

Generally, the MPC consists of a state estimator and a constrained regulator, therefore for state estimation, a Kalman filter is incorporated. The computation of the coefficients is done off-line and the Discrete-time Algebraic Ricatti Equation (DARE) is used to obtain the stationary one-step ahead state error covariance matrix. The static Kalman filter is utilized to estimate the current state from the filtered part while the predictions part is used by the constrained regulator which is also known as an Optimal Control Problem (OCP) to predict the future output trajectory given an input trajectory. The objective of the OCP consists of tracking error term that penalizes deviations of the predicted outputs from the setpoint and a regularization term that penalizes the changes in the inputs which is the manipulated variables. The resulting OCP which is represented as a Quadratic Programming (QP) is solved and the performance of MPC is demonstrated through simulations using MATLAB is presented.

The study shows that the static Kalman filter is well executed and the estimation of the current states and the prediction of the future output trajectory are accomplished. Subsequently, the performance of the MPC is investigated. In this study, the MPC is implemented to unconstrained and constrained MPC. The unconstrained MPC is implemented to evaluate a first-hand straightforward performance of MPC and from the demonstration, disturbances are compensated and new setpoint was tracked, except an abrupt peak is visible in the flow of input variables indicates that it is infeasible for real application.

The constrained MPC is formulated both for input and soft output constraints. When input constraint is introduced, the performance of MPC is exceptional although the transient response is slightly deprived, it is noticeable that the sharp peak in the flow of input variables is successfully suppressed, yielding a more relaxed flow. Whereas additional soft output constraint is included in the algorithm shows that the MPC operated as normal as the previous strategies without violating the input and output boundaries but the flow of the input variables is affected by a slight unsteady flow.

# Resumé

*to insert later*

# Acknowledgments

*In the name of Allah, the most Gracious, the most Merciful*
*So verily, with the hardship, there is relief (94:5)*
*This thesis is dedicated to my late grandfather,*
*Tan Sri Datuk Wira Haji Abdul Aziz Bin Tapa,*
*who had always believed in me.*

This PhD study would not have been possible without the help and support from the following which I am indebted and truly grateful for their presence throughout this challenging yet valuable experience journey, a journey which I evolved in many aspects.

Immense appreciation and deepest gratitude to Associate Professor John Bagterp Jørgensen, my research supervisor, for his expertise in the subject matter, constructive guidance, enthusiastic encouragement and useful critiques of this research work. Furthermore, I am grateful to have him as my supervisor for his advice and assistance in keeping my progress on schedule, for his patience and motivation.

Sincerely thankful for the scholarship granted by Universiti Teknikal Malaysia Melaka (UTeM) and Ministry of Education Malaysia (MoE) for funding both the studies and living expenses in Denmark.

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Nobody has been more important to me in the pursuit of this journey than the members of my family. I would like to thank my parents and my sister, whose love, prayers and encouragement are with me in whatever I pursue, to my parents-in-law for supporting and facilitating my way and to my friends for their cheers in hard times and fun in good times, and most importantly, to my pillars of strength, my husband Kamarul and our girl Sofea Nur for the unconditional love, understanding, guidance and endless inspiration.

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# List of figures


1.1 Schematic diagram of the modified quadruple tank process . . . . . . . . . . 4
1.2 Moving horizon implementation . . . . . . . . . . . . . . . . . . . . . . . . . . 5
1.3 Block diagram of the control structure for the MQT system process . . . . . 6

2.2 Schematic diagram of the modified quadruple tank process . . . . . . . . . . 11
2.3 Step responses of mass in each tanks for input $u_1$ and $u_2$ . . . . . . . . . . . 14
2.4 Step responses of outflow in each tanks for input $u_1$ and $u_2$ . . . . . . . . . . 15
2.5 LowNoise . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 16
2.6 LowNoise . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 18
2.7 MedNoise . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19
2.8 MedNoise . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20
2.9 The normalized step responses of the estimated transfer function with 10% step increment of input . . . . . . . . . . . . . . . . . . . . . . . . . . . . 21
2.12 Hankel Singular Values with $N = 20$ . . . . . . . . . . . . . . . . . . . . . . 23
2.13 MedNoise . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
2.15 Operating window with $\gamma_i$ in the RHP . . . . . . . . . . . . . . . . . . . . 29
2.16 Operating window with $\gamma_i$ in the LHP . . . . . . . . . . . . . . . . . . . . 30

3.1 Block diagram of the control structure for the MQT system process . . . . . 34
3.2 Disturbance Step Input $F_3$ and $F_4$ for **Condition 1** and **Condition 2** respectively . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 37
3.3 Kalman filter with measurement variable $h_1$ and $h_2$ with input disturbance of $F_3$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 38
3.4 Kalman filter with measurement variable $h_1$ and $h_2$ with input disturbance of $F_4$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 38

5.1 Flowchart of the simulation work for MPC implementation . . . . . . . . . . 50
5.2 15% step input change in reference trajectory of $F_1$ and $F_2$ for **Experiment 1** and **Experiment 2** . . . . . . . . . . . . . . . . . . . . . . . . . . 50
5.3 No step input change in $F_3$ and $F_4$ for **Experiment 1** . . . . . . . . . . . . 51
5.4 15% step input change in $F_3$ and $F_4$ for **Experiment 2** . . . . . . . . . . . 51
5.5 Unconstrained MPC for **Experiment 1** . . . . . . . . . . . . . . . . . . . . . 53
5.6 Unconstrained MPC for **Experiment 2** . . . . . . . . . . . . . . . . . . . . . 53
5.7 Input constrained MPC for **Experiment 1** . . . . . . . . . . . . . . . . . . . 55
5.8 Input constrained MPC for **Experiment 2** . . . . . . . . . . . . . . . . . . . 55

5.9 Input and soft output constrained MPC for **Experiment 1** . . . . . . . . . . 57
5.10 Input and soft output constrained MPC for **Experiment 2** . . . . . . . . . . 57

# Contents

**Preface** i

**Abstract** iii

**Resumé** v

**Acknowledgments** vii

**List of figures** ix

## I Summary Reports 1

**1 Introduction** 3
- 1.1 Modified Quadruple Tank System . . . 3
- 1.2 Model Predictive Control . . . 4
- 1.3 Thesis Objectives . . . 6
- 1.4 Thesis Organization . . . 7

**2 Modeling of a Modified Quadruple Tank System** 9
- 2.1 The Control Structure . . . 9
- 2.2 Non-linear Model . . . 9
  - 2.2.1 Deterministic Non-Linear Model . . . 10
  - 2.2.2 Stochastic Non-linear Model . . . 12
- 2.3 Non-linear Simulation Step Responses . . . 13
  - 2.3.1 Steady State of the MQT System . . . 13
  - 2.3.2 MQT System Operations . . . 14
  - 2.3.3 Step Reponses for Deterministic Model . . . 15
  - 2.3.4 Step Reponses for Stochastic Model . . . 17
  - 2.3.5 Transfer Function Identification . . . 17
- 2.4 Linear Discrete-time State Space Model . . . 23
  - 2.4.1 Linear System Realization . . . 25
  - 2.4.2 Discretization of a Linear System . . . 26
  - 2.4.3 Linear Discrete-time State Space Representation . . . 27

2.5 Operating Window in Steady State . . . . . 28
2.6 Summary . . . . . 32

**3 State Estimation** **33**
3.1 Discrete-time Linear System for State Estimation . . . . . 33
3.2 Kalman Filter . . . . . 34
3.3 Discrete-time Kalman Filter . . . . . 35
3.4 Static Kalman Filter . . . . . 36
3.4.1 Simulation for the Static Kalman Filter . . . . . 36
3.5 Offset-free Control . . . . . 39
3.5.1 Input Disturbance Model . . . . . 39
3.6 Summary . . . . . 40

**4 Model Predictive Control** **41**
4.1 Unconstrained Model Predictive Controller . . . . . 41
4.2 Constrained Model Predictive Controller . . . . . 44
4.2.1 Input Constrained MPC . . . . . 44
4.2.2 Input and Soft Output Constrained MPC . . . . . 45
4.3 Summary . . . . . 47

**5 Simulations and Analysis** **49**
5.1 Overview . . . . . 49
5.2 Closed-loop Simulation for Unconstrained MPC . . . . . 52
5.3 Closed-loop Simulation for Constrained MPC . . . . . 54
5.3.1 Input Constrained MPC . . . . . 54
5.3.2 Input and Soft Output Constrained MPC . . . . . 56
5.4 Summary . . . . . 58

**6 Conclusion** **59**
6.1 Modeling of Modified Quadruple Tank System . . . . . 59
6.2 State Estimation for the Discrete-Time Linear System . . . . . 60
6.3 Development and Simulation of Model Predictive Control . . . . . 60

**Bibliography** **63**

# II Published Journal and Conference Papers 67

**A Modeling and simulation of a modified quadruple tank system** **69**

**B Linear discrete-time state space realization of a modified quadruple tank system with state estimation using Kalman Filter** **77**

**C Model based control implementation of a modified quadruple tank system using a predictive control strategy 91**

# Part I

# Summary Reports

CHAPTER 1

# Introduction

In this chapter, the description of the Modified Quadruple Tank System is presented in detail. It illustrates the importance of the Model Predictive Control implementation for the system. We featured the main objective of the research project and the explanation of the thesis organization.

## 1.1 Modified Quadruple Tank System

The modified quadruple tank (MQT) system as shown in Figure 1.1 is a modification of the four tank system inspired by [1]. Compared to the original four-tank system, two unknown disturbance is added to the MQT system entering the top tanks. The MQT system is an exemplification of a multi-input-multi-output (MIMO) system, an illustration of the real-world complex system applications which is widely used for education in modelling and demonstrating advanced control strategies [2], [3]. Most of the real application control tasks in the industry handle mostly non-linear systems, affected by multiple inputs and outputs that have complicated variables interactions and significant uncertainties. These unique and complex interactions make the MQT system a good example to demonstrate the modelling and controller application study as discussed in [4–6]. It has immeasurable disturbance variables, significant cross binding parameters which cause unwanted output disturbance when defining the control input in order to have desired output and needs linearization due to its non-linearity, which causes further errors into the control loop [7]. Moreover, due to the modification, it is also affected by unknown measurement noise and disturbance variables that are considered stochastic [8]. Therefore, it is important to study the dynamic behaviour of the system and its potentials before involving any controller.

The MQT system is a simple process, consist of four identical tanks and two pumping system but yet illustrates a system that is non-linear, MIMO and complicated interactions between manipulated and controlled variables. The states $x$ of the modified quadruple tank system are the masses of water in different tanks as the states of the masses changes over time due to the dynamics of the water flow in and out of each tank. The main objective of this lab-scale system is to measure and control the levels of water in the lower tanks (Tank 1 and 2) to some desired set points by manipulating the flow rates $F_1$ and $F_2$ which are distributed across all four tanks, represents the dynamics of multi-variable interaction since each manipulated variables influences the outputs. Therefore the height of the water level in these two tanks, $h_1$ and $h_2$ is measured and controlled. Usually, the controlled variables is a subset of the measured variable, $y$. The pumping system directs a fixed fraction of $F_1$ and $F_2$, denoted as $\gamma_1$ that distribute the water to Tank 1 and Tank 4, and $\gamma_2$ for Tank 2 and Tank 3 at a rate of $q_{i,in}$ $i \in \{1, 2, 3, 4\}$ respectively [9]. The values for $\gamma_1$ and $\gamma_2$ differs for minimum phase (RHP) and non-minimum phase (LHP). The sensors measure the height of water level in each tank,

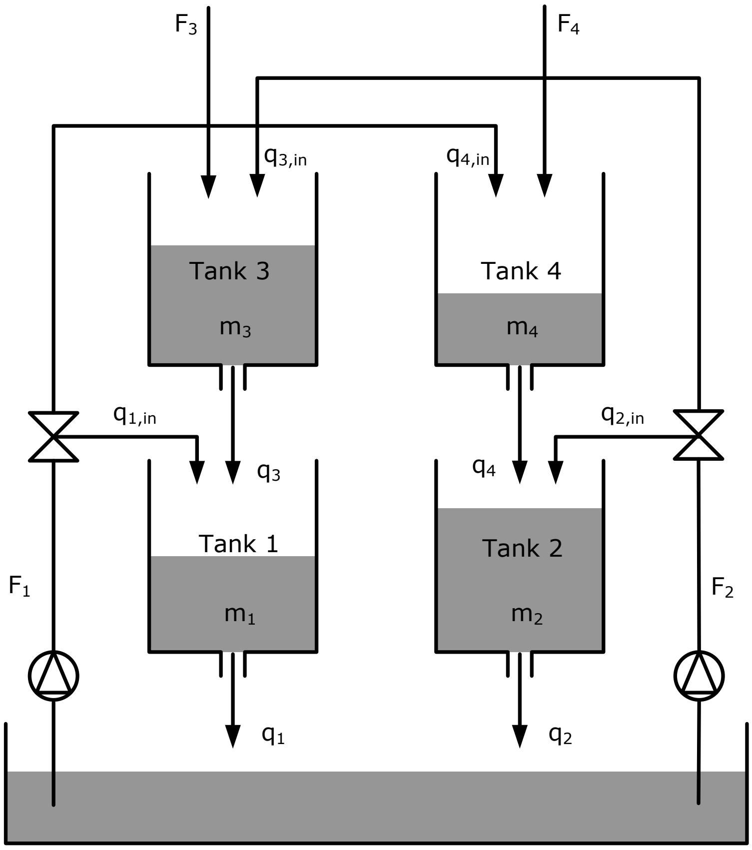



**Figure 1.1:** Schematic diagram of the modified quadruple tank process

$h_i, i \in \{1, 2, 3, 4\}$ thus, the measured variable is affected by the noise from these sensors and due to this condition, the measured variables, $y$ consists of the actual values of the height and sensor noise. Whereas liquid is added to Tank 3 and Tank 4 resembling external disturbances $d$ and the flows denoted $F_3$ and $F_4$ are unmeasured unknown disturbances and stochastic variables that is assumed to be normally distributed, hence cannot be controlled. However, for the modelling purposes in the following chapter, we assumed no noise occurrence and impeccable measurements.

The MQT system will be fully utilized throughout this work to assimilate the fundamental theory of the MPC and further described in the following section. While [5, 6, 10] has the opportunity to work with the actual pilot plant, this work is done in a simulation environment. Without an actual process plant, an accurate first-principles model can be achieved [11].

## 1.2 Model Predictive Control

A model-based controller is one of the advanced control strategy that is currently common and extensively recognized in industry and academic, famously known as Model

Predictive Control (MPC). The MPC first breakthrough is from a seminal publication of Model Predictive Heuristic Control [12] and later [13] who came out with Dynamic Matrix Control.

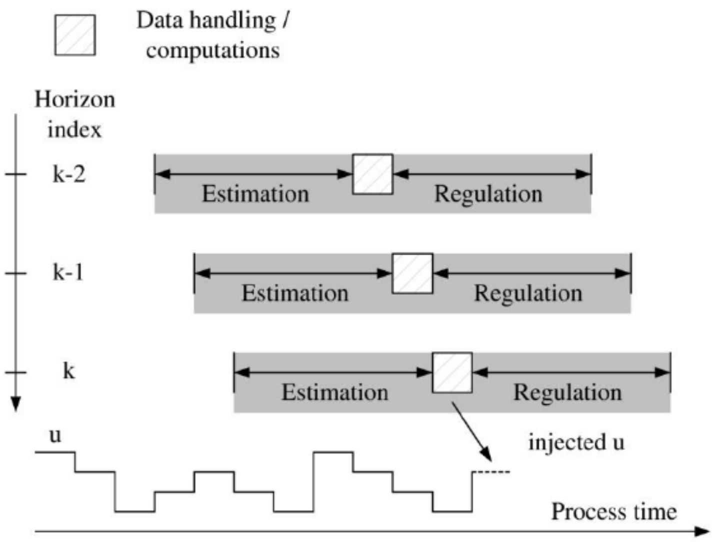



**Figure 1.2:** Moving horizon implementation

MPC is a controller that utilizes the identified model of a system to predict its future behaviour, given a prediction horizon. The main idea is to minimize the cost function and taking into account the constraints. Then the first controller moves are implemented at a sampling instants over the control horizon, by implementing only the first move the optimal feedback is achieved and then the complete sequence will be repeated again, which is known as moving horizon concept [11]. Figure 1.2 shows an illustration of the concept. This flexibility is one of the important significant advantages of MPC [14]. Nowadays the applications of MPC are not limited to the process control field, but also including other various fields such as automotive, energy management, medical and economy.

This PhD study is emphasising on implementing the predictive control strategy on a lab scale system that exhibits the characteristic of a complex MIMO system, the modified quadruple tank system which is briefly described in the previous section to comprehend the fundamental theory of MPC and develop the procedure on the application of the control strategy by simulations.

Several strategies of controllers are implemented on the quadruple tank system such as [15] and [16], while [17] and [18] has been extensively described the application of MPC on the quadruple-tank system with different approaches. A comparative study of model-based control for the four-tank system using IMC and DMC is provided by [16] and a year later an analysis of robust control for the identical system is done [19]. The Kalman filter is incorporated for state estimation to obtain an optimal MPC as shown in [20] and specified in detail in [21] attained from the Discrete-time Algebraic Riccati Equation (DARE). Offset-free MPC for a single tank is explained in [22] but an off-set free MPC caused by the model's mismatch is proposed by [23] but for a different orientation of quadruple tank system and by considering only the state disturbance.

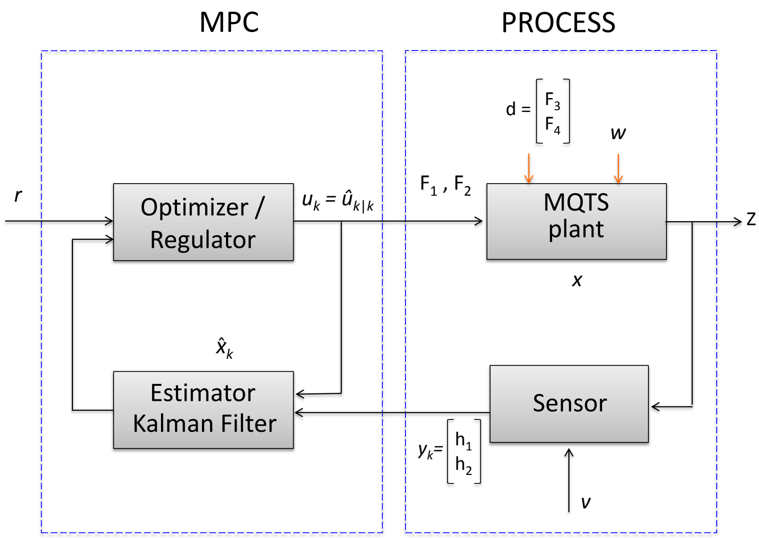



**Figure 1.3:** Block diagram of the control structure for the MQT system process

One of MPCs advantages is the ability to work within certain constraints, therefore the main focus of this work is to demonstrate the implementation of unconstrained and input constrained MPCs complete with the derivation of the equations, as reported in Paper C. An addition to that, we resume the study by considering both input and soft output constraints MPC in a single problem. To solve the Optimal Control Problem (OCP), we express the control task which is tracking of the setpoint trajectory as a quadratic optimization problem by performing several experiments and simulations studies for observation to demonstrate the versatility of the advanced controller.

Referring to the block diagram of the control structure for the MQT system in Fig.1.3, the model predictive control part consists of an estimator which estimates the current states of the system, $\hat{x}$ given the previous measurement from the process, $y$ and a regulator which minimizes the difference between the reference values, $r$ and the controlled variables, $z$ with respect to the manipulated variables, $u$. The output of the regulator which is the new input variables is then fed to the MQT system thus yielding a new state and a new measurement. This process is iteratively repeated for a specified timespan or until a certain stopping criterion is reached in a closed loop manner. From the diagram, it is possible to visualise the implementation of the MPC that will yield the optimum input to the system which would result in an approach to the desired reference values.

## 1.3 Thesis Objectives

The main objective of this thesis is to develop the procedure of an advanced model-based controller implementation and to demonstrates the procedure on a lab-scale industrial process, using linear MPC. In order to achieve the main objective of the thesis, it can be subdivided into several topics and multiple objectives need to be accomplished.

1. **Modeling of the Modified Quadruple Tank System.**

   We obtain first-principles engineering model of a Modified Quadruple Tank System by describing the non-linear dynamics of the system in which the equations are derived from the fundamental physical processes. The conservation of mass is applied to develop a simple model and we simulate the system which described by Ordinary Differential Equations (ODEs). The system is fully described and all important variables are outlined. Then, the non-linear continuous state space model of the system is transformed to a linear discrete-time state space model through linearization and discretization. In order to rewrite the difference equation system representation in a more structured form, the Markov Parameters is introduced.

2. **State Estimation for the Discrete-Time Linear System.**

   We want to estimates all the variables which represent the internal conditions or the status of the system at a specific given time so as to allow for future output prediction and to design the control algorithms. A Kalman's state estimator for a non-linear multivariable process using a linear state-space model is solved by the algebraic Ricatti equation to estimates the current state of the system and the future output predictions.

3. **Development and Simulation of Model Predictive Control.**

   We develop and demonstrate the implementation of Model Predictive Control for the Modified Quadruple Tank System by simulation. The predictions part of the Kalman filter is used by constrained regulator, an optimal control problem to predict the future output trajectory, the results that are represented as a quadratic problem is solved and the performance of the controller is demonstrated through simulations.

## 1.4 Thesis Organization

This PhD thesis is written based on a number of scientific articles published in peer-reviewed international conference proceedings and a scientific journal. The thesis is divided into two parts, Part I and Part II.

Part I is the summary report comprised of several chapters which start with Chapter 1 outlines the background and aim of the research work followed by the detail explanation of the modelling part of the system in Chapter 2. Chapter 3 introduces the concept of state estimation where the Kalman filter is incorporated and provides a description of the disturbance model. Chapter 4 demonstrates the implementation of MPC strategy while Chapter 5 presents the results and analysis of the controller implementation. Finally, Chapter 6 concludes the study with a summary of the method used and the results. In Part II three research papers are included. These papers are published and submitted during the project period.

# CHAPTER 2
# Modeling of a Modified Quadruple Tank System

In this chapter we describe the modified quadruple tank system with all the important parameters are selected and variables are defined. We want to obtain an accurate first-principles model of the process by describing the dynamics of the system in which the equations are derived from the fundamental physical processes. A non-linear ordinary differential equation (ODE) model is obtained and we reconstruct the model in an appropriate form for future model-based controller design, a form that is suitable for computational operation and analysis purposes. This chapter provides a summary of the scientific dissemination in Papers A and B.

## 2.1 The Control Structure

Previously in section 1.1, the dynamics of the MQT system is described and the control structure of the system is outlined. To address the problem of controlling the tanks system we first identify all the variables.

Let $x$ indicates the state variables signifies the water levels in tanks, $y$ is the measured variables indicates the water level in the tank, $u$ indicates the manipulated variables (MVs) or inputs, $z$ is the controlled variables (CVs) represents the tanks which we wish to achieve the desired set points and $d$ is the disturbances. This can be written as

$$\mathbf{x} = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \end{bmatrix}^T \tag{2.1a}$$

$$\mathbf{y} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 \end{bmatrix}^T \tag{2.1b}$$

$$\mathbf{u} = \begin{bmatrix} F_1 & F_2 \end{bmatrix}^T \tag{2.1c}$$

$$\mathbf{d} = \begin{bmatrix} F_3 & F_4 \end{bmatrix}^T \tag{2.1d}$$

$$\mathbf{z} = \begin{bmatrix} h_1 & h_2 \end{bmatrix}^T \tag{2.1e}$$

In the following sections, two non-linear continuous time models will be described based on the parameters that determine the dynamics and the response of the system. All the parameter values of the modified quadruple tanks system are shown in Table 2.1.

## 2.2 Non-linear Model

In this section, first, we develop a deterministic model and then followed by a stochastic model to represent a more realistic model by including the process and measurement

**Table 2.1:** Parameter Values

| **Description** | **Symbol** | **Value** | **Unit** |
|---|---|---|---|
| Tank cross sectional area | $A_i$ | 380.1327 | $cm^2$ |
| Pipe cross sectional area | $a_i$ | 1.2272 | $cm^2$ |
| Acceleration of gravity | $g$ | 981 | $cm/s^2$ |
| Density of water | $\rho$ | 1.00 | $g/cm^3$ |
| Valve distribution constants | $\gamma_1$ | 0.45 | |
| Valve distribution constants | $\gamma_2$ | 0.40 | |

noises. Mathematical models for the sensors and outputs are also derived and included. Then the non-linear simulation step responses are presented to show the results and analysis from the modelling part of the work. From the simulation results, the transfer function is identified and the noise is estimated. Finally, a steady state analysis is investigated to obtain the operating window for opting set points.

### 2.2.1 Deterministic Non-Linear Model

The complete deterministic non-linear model can be expressed by a set of first-order differential equations in the form of

$$\dot{\mathbf{x}} = f(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t), p) \tag{2.2a}$$
$$\mathbf{y}(t) = \mathbf{g}(x(t)) \tag{2.2b}$$
$$\mathbf{z}(t) = \mathbf{h}(x(t)) \tag{2.2c}$$

with $p$ is the parameters and initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{2.3}$$

The measurement from the sensor is modelled as linear function is defined as $y(t)$ and the output function is defined as $z(t)$. $g(x(t))$ and $h(x(t))$ are the measurement and output functions related to the process state. Then we apply the conservation of mass for each tank to formulate the differential equations and we simulate the system which described by Ordinary Differential Equations (ODEs) in Matlab [24] as

$$\frac{d}{dt}m_1(t) = \rho(q_{1,in}(t) + q_3(t) - q_1(t)) \tag{2.4a}$$
$$\frac{d}{dt}m_2(t) = \rho(q_{2,in}(t) + q_4(t) - q_2(t)) \tag{2.4b}$$
$$\frac{d}{dt}m_3(t) = \rho(q_{3,in}(t) + F_3(t) - q_3(t)) \tag{2.4c}$$
$$\frac{d}{dt}m_4(t) = \rho(q_{4,in}(t) + F_4(t) - q_4(t)) \tag{2.4d}$$

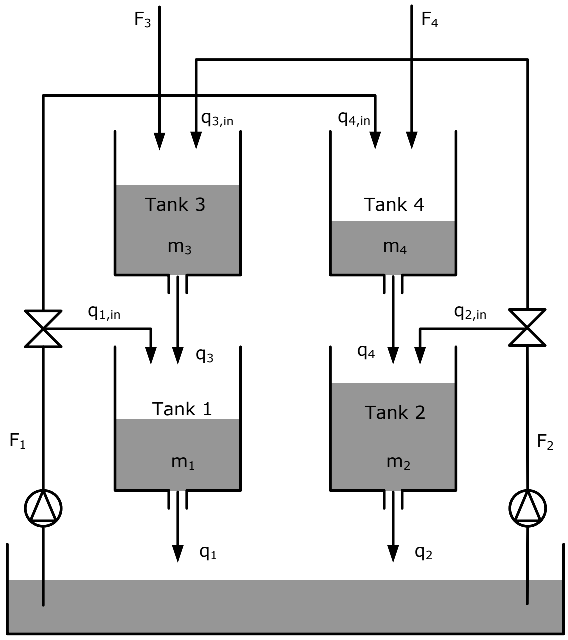



**Figure 2.2:** Schematic diagram of the modified quadruple tank process

The mass balances (2.4) constitute the differential equations in the model describing the states of the system. The initial values for the mass of water in each tank at time $t_0$ are

$$m_i(t_0) = m_{i,0} \qquad i = 1, 2, 3, 4 \tag{2.5}$$

The static mass balance of the two valves is used to obtain the inflow rates from the valves for individual tanks. Here $\gamma_1$ and $\gamma_2$ are the flow distributions constants for valves 1 and 2 respectively. Besides, $F_1(t)$ and $F_2(t)$ are the input flow rates coming from these pumps.

$$q_{1,in}(t) = \gamma_1 F_1(t) \tag{2.6a}$$
$$q_{2,in}(t) = \gamma_2 F_2(t) \tag{2.6b}$$
$$q_{3,in}(t) = (1 - \gamma_2) F_2(t) \tag{2.6c}$$
$$q_{4,in}(t) = (1 - \gamma_1) F_1(t) \tag{2.6d}$$

The height of the liquid and the flow rates out of each tank is calculated in this section. Let the measurements and outputs be the heights, $h_i$. The height is calculated by the relations of mass and volume of the water in each tank where mass $m_i$ in tank $i$ is given by

$$m_i = \rho V_i \qquad i = 1, 2, 3, 4 \tag{2.7}$$

where $\rho$ is the density of the fluid and let $V_i$ be the volume of the water in all the four tanks with an assumption that $A_i$ is the cross-sectional area for each tank

$$V_i = A_i h_i \qquad i = 1, 2, 3, 4 \tag{2.8}$$

the height $h_i$ of the liquid level in tank $i$ is calculated by the relations

$$h_i = \beta_i m_i \qquad i = 1, 2, 3, 4 \tag{2.9}$$

with $\beta_i$ is

$$\beta_i = \frac{1}{\rho A_i} \qquad i = 1, 2, 3, 4 \tag{2.10}$$

The flow rates of the liquid $q_i(t)$ is calculated by applying Bernoulli's Principle to each tank which gives the following volumetric outflow rate

$$q_i(t) = a_i \sqrt{2gh_i(t)} \qquad i = 1, 2, 3, 4 \tag{2.11}$$

where $a_i$ is the cross section of the pipes, $g$ is the gravity, $h_i$ is as given as (2.9) and $\alpha_i$ is

$$\alpha_i = a_i \sqrt{2g\beta_i} \qquad i = 1, 2, 3, 4 \tag{2.12}$$

the volumetric flow rate in the outlet pipes from the tanks are

$$q_i(t) = \alpha_i \sqrt{m_i} \qquad i = 1, 2, 3, 4 \tag{2.13}$$

### 2.2.2 Stochastic Non-linear Model

In this section both process noise and measurement noise is included in the model of the system, (2.2a) and measured variables, (2.2b) respectively. This is because, in practice, the system can be affected by noise and disturbance, uncontrolled disturbance variables change in time and the uncertainty of the measurements can't be removed completely. Therefore in compensation for this, we include a normally distributed measurement error and make the disturbances vary randomly. Specifically, an additional white-noise is introduced on the measurements and let the uncontrolled disturbance flows change as piecewise constants between samples with a similar normally distributed disturbance. The system in equation 2.2 have a new measurement equation and stochastic disturbance. The new stochastic disturbance become

$$d(t) = d_k \quad \text{for} \quad t_k < t < t_{k+1} \tag{2.14}$$

where $d(t)$ is piecewise constant. In this case, equation (2.14) have the stochastic component

$$d_k = \begin{bmatrix} F_3 \\ F_4 \end{bmatrix} + w_k \tag{2.15}$$

where $w_k$ is the process noise that is stochastic and assumed to be normally distributed, $w_k \sim N(0, Q)$, due to the unknown information regarding the distribution and this results in the non-linear model.

Let the measurement noise, $\mathbf{v}(t)$ signifies the occurrence of noise from the sensors during the process of measuring in each tank and it is normally distributed, $\mathbf{v}(t) \sim N(0, R_v)$. $\mathbf{v}(t)$ is being added to the measured variables, (2.2b) giving

$$\mathbf{y}(t) = g(\mathbf{x}(t)) + \mathbf{v}(t) \tag{2.16}$$

## 2.3 Non-linear Simulation Step Responses

In this section, the non-linear system is analysed where the step responses of the non-linear simulation are evaluated. From these responses, we will identify the continuous-time transfer function of the MQT system and it is possible to characterise the system for which the dynamics are unknown.

Now that the deterministic and stochastic model is obtained, we visualized the behaviour of the system and verify the model by simulations. This is done so to have the basic understanding of the MQT system operation and the directions of the water flow, step input test is applied and the output response is observed.

Then the output responses of the deterministic non-linear system are presented, discussed and analysed. Using the non-linear deterministic model we simulate the system with three level of input increment resulting in the step responses and normalized step responses. The idea is to analyse the response in the measurements, $y$ by manipulating the inputs, $u$ by referring them to the steady-state values, $u_{ss}$ in order to define the step responses. The same simulation is done for the non-linear stochastic model with not only the input increment but including measurement error for respectively low, medium and high variance of the noise.

In order to identify the continuous time transfer function of the MIMO system, the normalized step responses is used by estimating the characteristic parameters. The identified transfer function is utilized to estimate the mean and variance of the injected noise and to run an analysis of the Markov parameters for the purpose of determining the sampling time for further use.

### 2.3.1 Steady State of the MQT System

During steady state, $x_{ss}$ the system is at a fix point where the states remain constant in time where $\dot{\mathbf{x}} = 0$, the root of the RHS of function 2.2a, giving

$$f(x_{ss}, u_{ss}, d_{ss}) = 0 \tag{2.17}$$

which equations 2.4 becomes a non-linear of four equations with four unknown and let the steady-state values for the flow rates be

$$u_{ss} = \begin{bmatrix} 300 \\ 300 \end{bmatrix} \qquad d_{ss} = \begin{bmatrix} 250 \\ 250 \end{bmatrix} \tag{2.18}$$

The system which is given by function (2.17) is solved using a Matlab command, **fsolve** with a wrap function since the function of the model is autonomous and can't be inserted directly in the **fsolve** command. This is because the **fsolve** command works with a function whose first argument is the variable of the system i.e $x$ instead of $t$. The corresponding measurements, $y_{ss}$ and output, $z_{ss}$ is solved using sensor and output functions respectively. The calculated steady state values obtained is as shown in the matrices below

$$x_{ss} = \begin{bmatrix} 4.1068 \\ 3.6822 \\ 2.3787 \\ 2.2157 \end{bmatrix} 10^4 [g] \qquad d_{ss} = \begin{bmatrix} 108.0357 \\ 96.8675 \\ 62.5759 \\ 58.2863 \end{bmatrix} [cm] \qquad z_{ss} = \begin{bmatrix} 108.0357 \\ 96.8675 \end{bmatrix} [cm] \tag{2.19}$$

### 2.3.2 MQT System Operations

The first part of the simulation is for the deterministic model where the response of the system is perturbed by an input in one of the input variables, $u_1 = F_1$ and $u_2 = F_2$. The simulation is started when $t = t_0$ during steady state and solved using **ode15**, a Matlab command up to $t = t_f$, where the time span for the integration/iteration can be determined. For this simulation, $N = 100$ points that is uniformly distributed form $t_0$ until $t_f$ is predetermined.

The responses is exhibited in Fig.2.3 and Fig.2.4 for mass, $m$ and outflow, $Q_{out}$ respectively. On the left of the panels shows the responses for $u_1$ input and on the right is the responses for $u_2$ input. The system reacts to the inputs and illustrates the dynamic of the water flow between connected tanks. As for the first input, $F_1$ results in deviations of the mass of $m_1$, $m_2$ and $m_4$ which represents the operation of the first pumping system. The operation of the second pumping system can be seen as in the right of the panels where input in $F_2$ does not give any effects to the mass of $m_4$ but we can see the changes of mass accordingly in the other tanks. This dynamics can also be seen in the outflow responses in Fig.2.4.

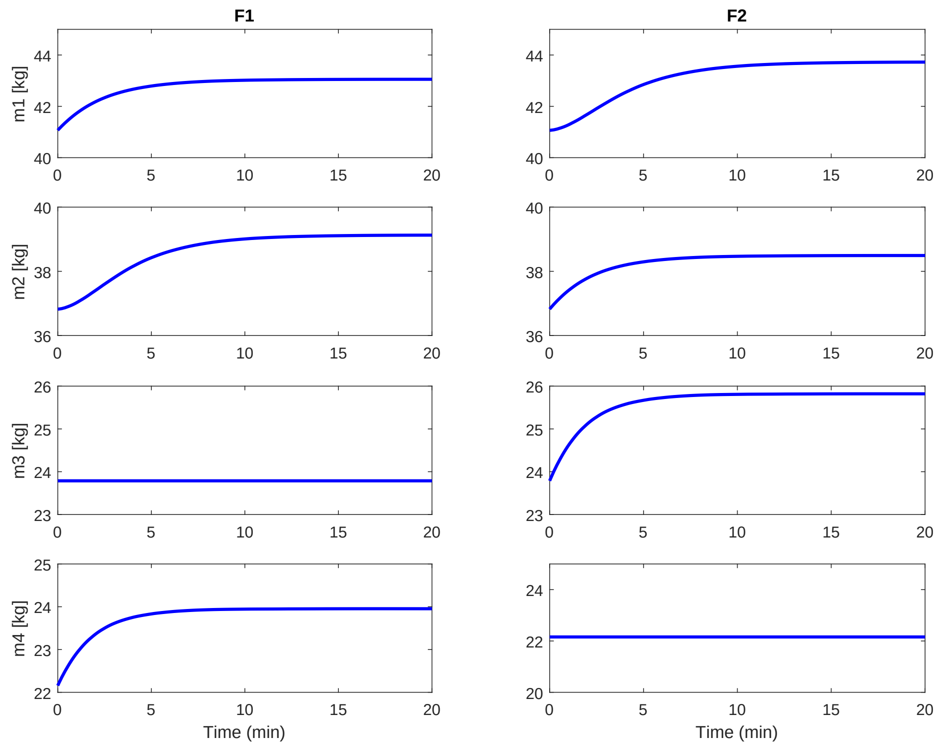



**Figure 2.3:** Step responses of mass in each tanks for input $u_1$ and $u_2$

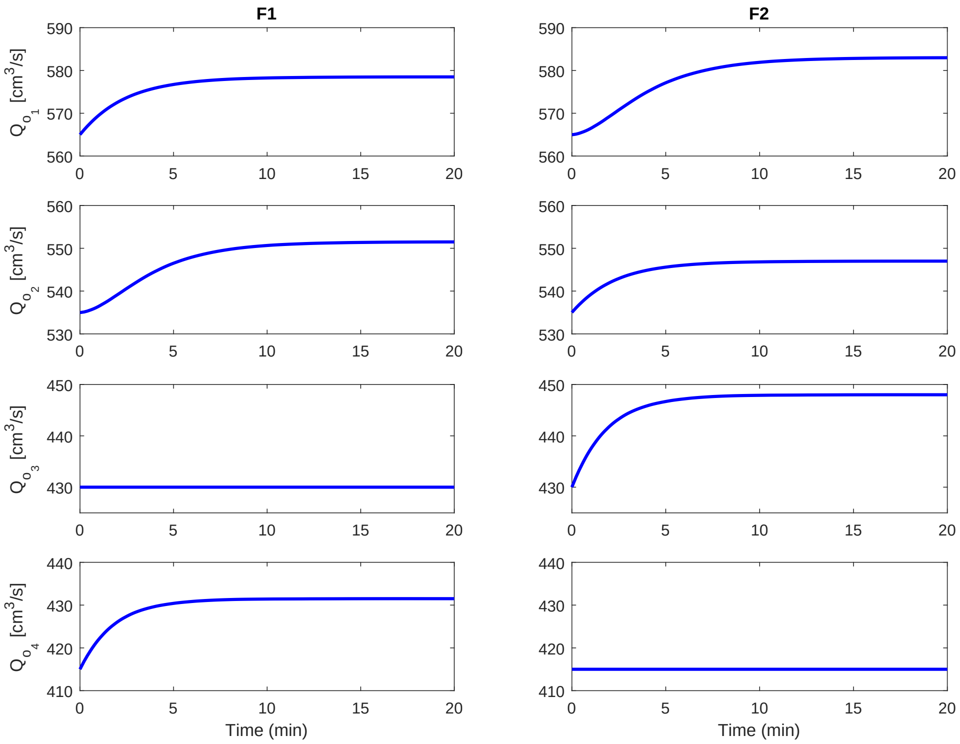



**Figure 2.4:** Step responses of outflow in each tanks for input $u_1$ and $u_2$

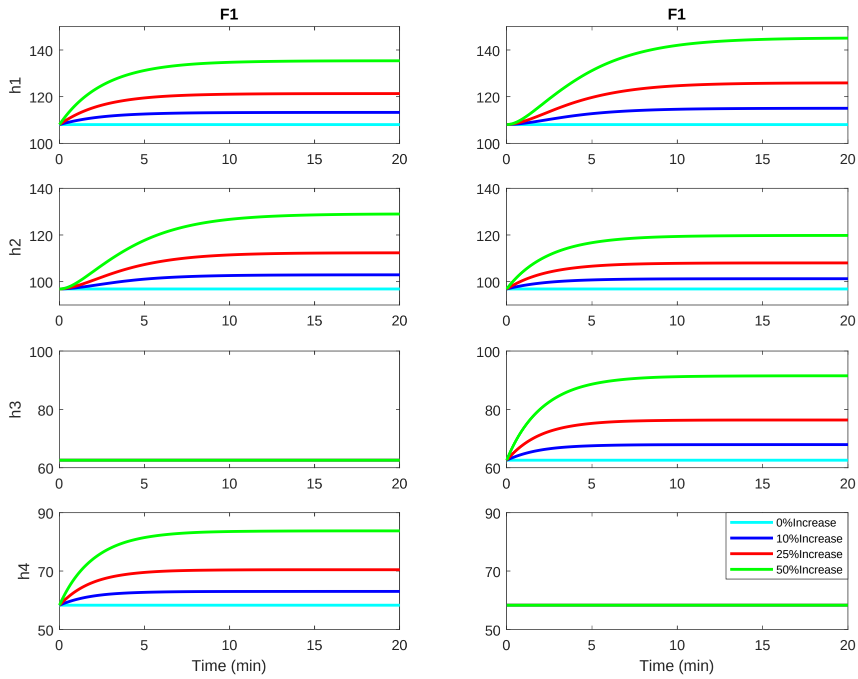
### 2.3.3 Step Reponses for Deterministic Model

With given input of $u_1$ and $u_2$, we want to analyse the level of fluid in each tank with different levels of steps increment. Fig.2.5a shows the responses of a 10% step increment in blue, 25% in red and 50% in green. In a glimpse, the order of transfer function for each and every output responses can be observed. Details of the transfer function identification form the output responses are described in the following section.

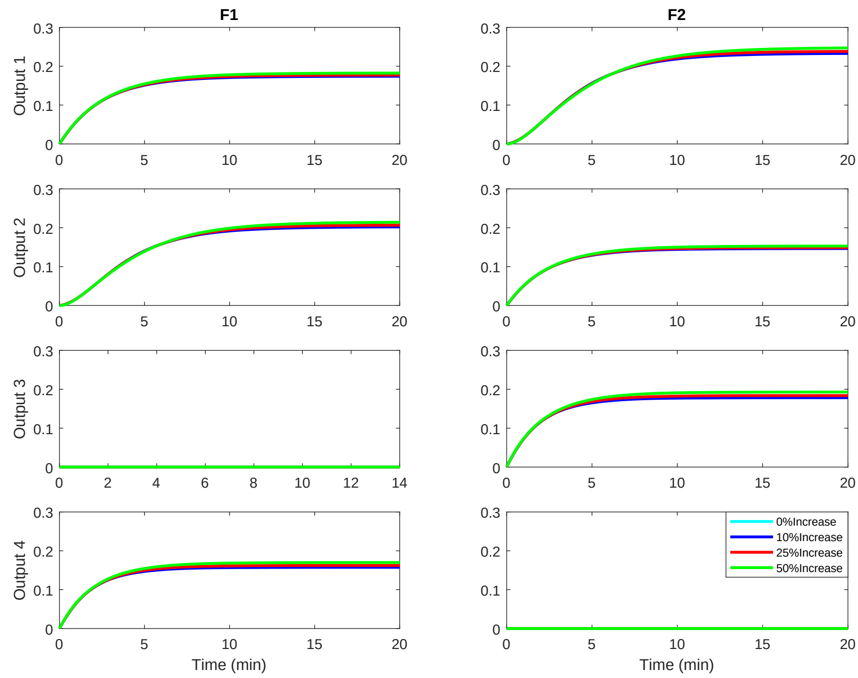
As for the normalised step responses, all of the responses are computed for each manipulated variable separately. Normalised step responses are defined as

$$S(t) = \frac{x(t) - x_{ss}}{u(t) - u_{ss}} \tag{2.20}$$

It is important to note that in equation 2.20, the numerator gives the step response in terms of the state variables which is the deviation from the steady state and the denominator provides the deviation of the input variables from the steady state which is the input step size. Thus, with a given deviation of inputs, $S(t)$ represents the deviation of the output from the steady state. Fig.2.5b shows the normalized step response for 10%, 25% and 50% increases in blue, red and green respectively for input $F_1$ on the left and input $F_2$ on the right respectively for the deterministic model without measurement noise. The differences between the step increases are insignificant but existent and this is




**(a)** Step responses




**(b)** Normalized step responses

**Figure 2.5:** Step responses and normalized step response for height of the tank liquid level in 10%, 25% and 50% increment of input $F_1$ and $F_2$

expected for the non-linear model, whereas the responses would be identical if the model is linear.

### 2.3.4 Step Reponses for Stochastic Model

The step responses have also been evaluated for the stochastic nonlinear model including only measurement noise ($v_t \sim N(0, R)$) for the same increment step sizes defined in the previous section. The stochastic non-linear step responses obtained have been computed for three different noise levels with variances as shown in the matrices below:

$$R_L = \begin{bmatrix} 1^2 & 0 & 0 & 0 \\ 0 & 1^2 & 0 & 0 \\ 0 & 0 & 1^2 & 0 \\ 0 & 0 & 0 & 1^2 \end{bmatrix} \qquad R_M = \begin{bmatrix} 2^2 & 0 & 0 & 0 \\ 0 & 2^2 & 0 & 0 \\ 0 & 0 & 2^2 & 0 \\ 0 & 0 & 0 & 2^2 \end{bmatrix} \qquad R_H = \begin{bmatrix} 3^2 & 0 & 0 & 0 \\ 0 & 3^2 & 0 & 0 \\ 0 & 0 & 3^2 & 0 \\ 0 & 0 & 0 & 3^2 \end{bmatrix}$$

Fig.2.6a, Fig.2.7a and Fig.2.8a presents the results for the step responses for stochastic non-linear model with 10% (blue), 25% (red) and 50% (black) increment in input steps and one for each with different level of noise. Whilst normalized step response for three different noise levels are presented in Fig.2.6b, Fig.2.7b and Fig.2.8b respectively. From observation, it can be seen that the noise is damped for the normalization constant for a bigger increment of input step. The blue line looks noisy compared to the black lines regardless of having the same level of noise.

### 2.3.5 Transfer Function Identification

In order to identify the transfer functions for the MQT system, the transfer function, $G$ is estimated from these normalized step responses from $U$ to $Y$

$$Y = GU \tag{2.21}$$

and the MIMO system can be represented as

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{2.22}$$

Transfer function $G(s)$ can be either first or second order SISO transfer function and the analytical forms can be written respectively as

$$G(s) = \frac{K}{\tau_1 + s}, \qquad S(t) = K(1 - e^{\frac{-t}{\tau_1}}) \tag{2.23a}$$

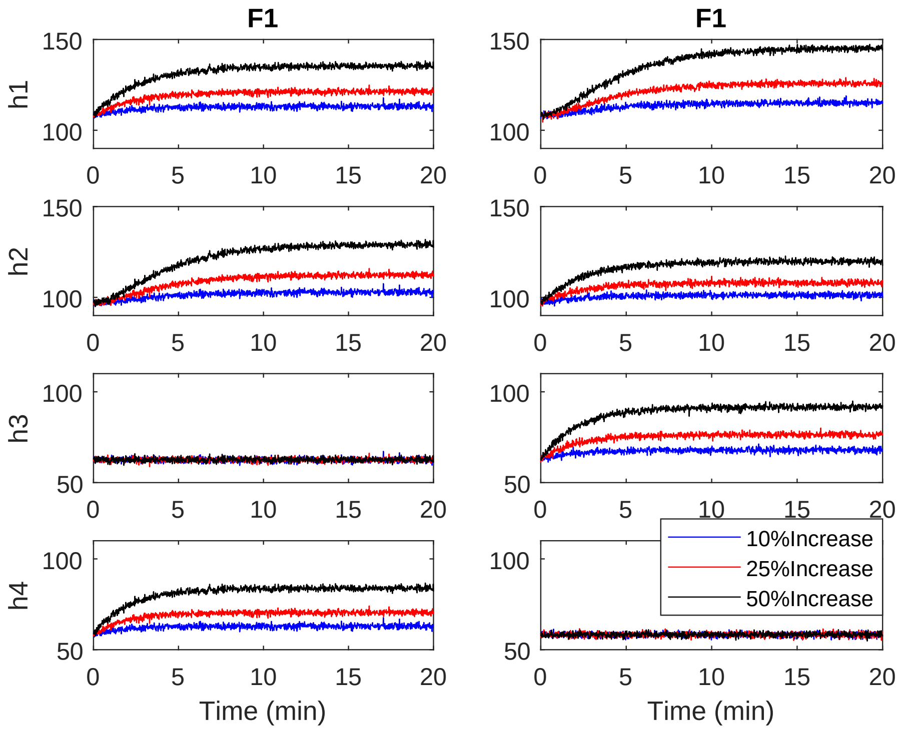
$$G(s) = \frac{K}{(\tau_1 + s)(\tau_2 s + 1)}, \qquad S(t) = K(1 - Ae^{\frac{-t}{\tau_1}} - Be^{\frac{-t}{\tau_2}}) \tag{2.23b}$$

where A and B are given as

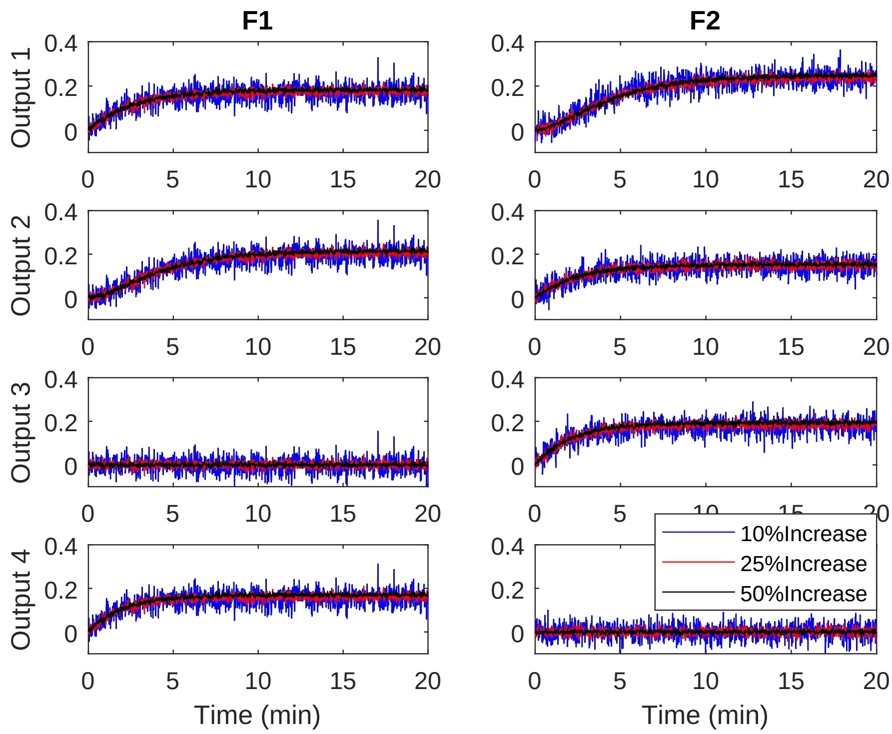
$$A = \frac{\tau_1}{\tau_1 - \tau_2} \tag{2.24a}$$

$$B = \frac{\tau_2}{\tau_2 - \tau_1} \tag{2.24b}$$

and $\tau_1$ and $\tau_2$ are the time constants of each SISO system and $K$ is the gain. To obtain these parameters from the normalized step responses of the non-linear deterministic

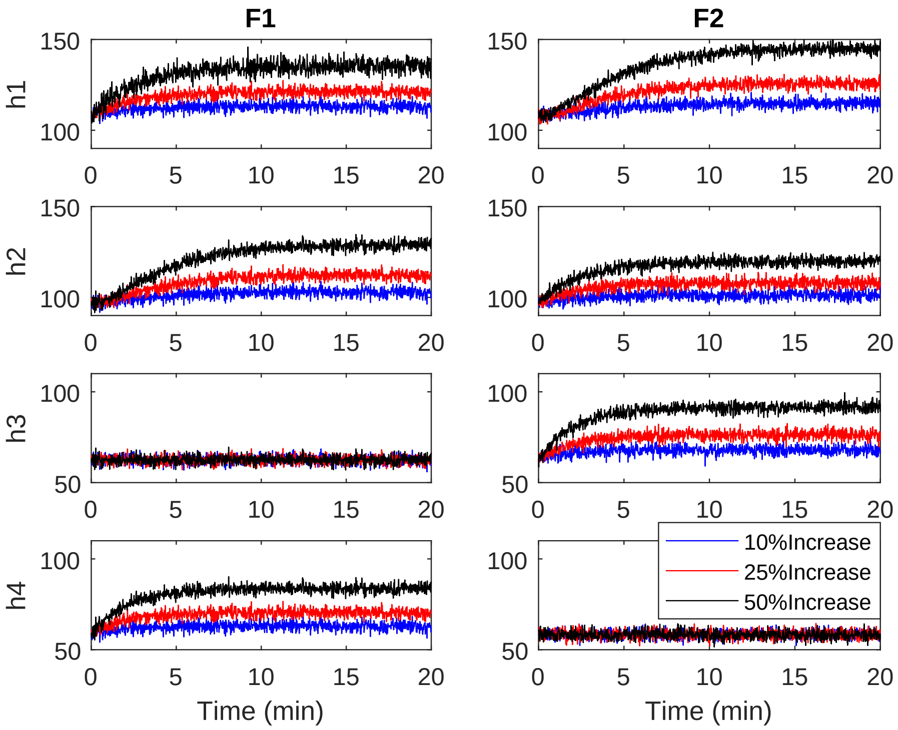



**(a)** Step responses for stochastic non-linear model

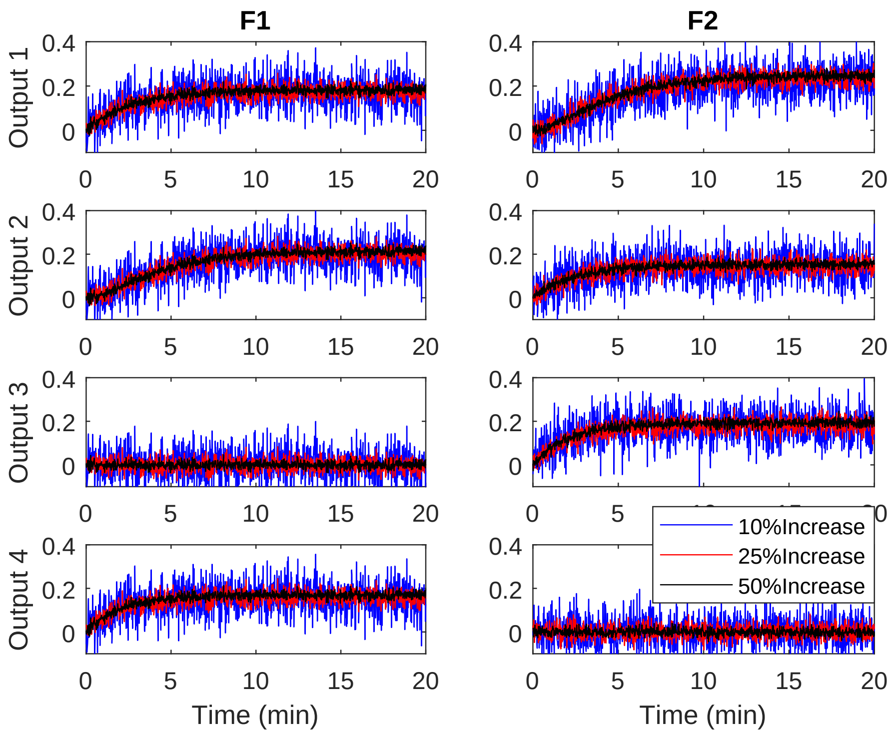



**(b)** Normalized step responses for stochastic non-linear model

**Figure 2.6:** Step responses and normalized step responses for stochastic non-linear model with 10%, 25% and 50% input increment and low noise level (variance 1)

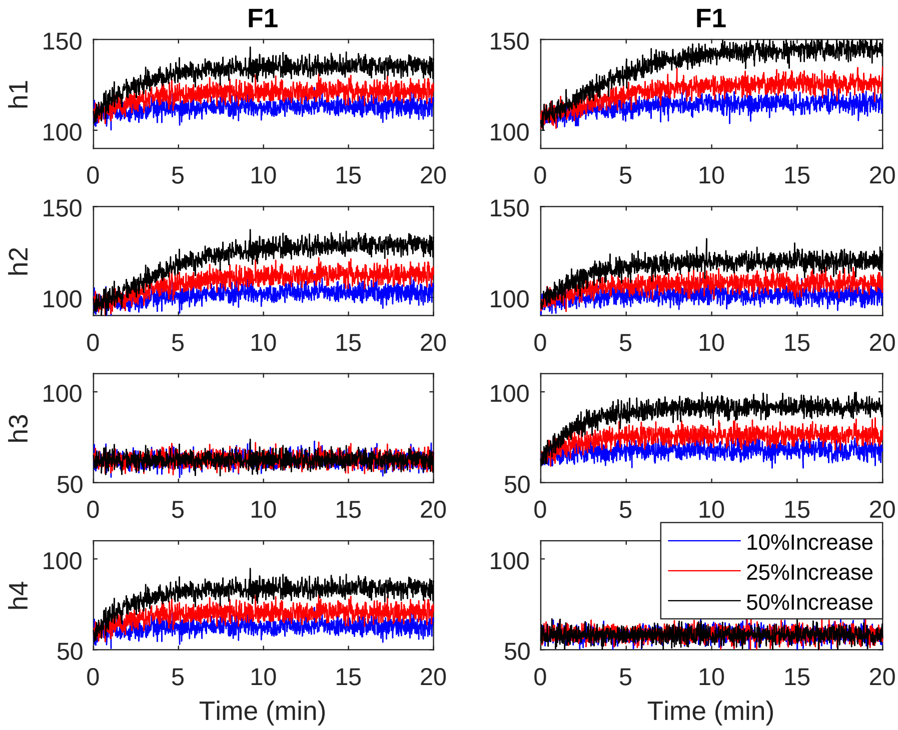



**(a)** Step responses for stochastic non-linear model

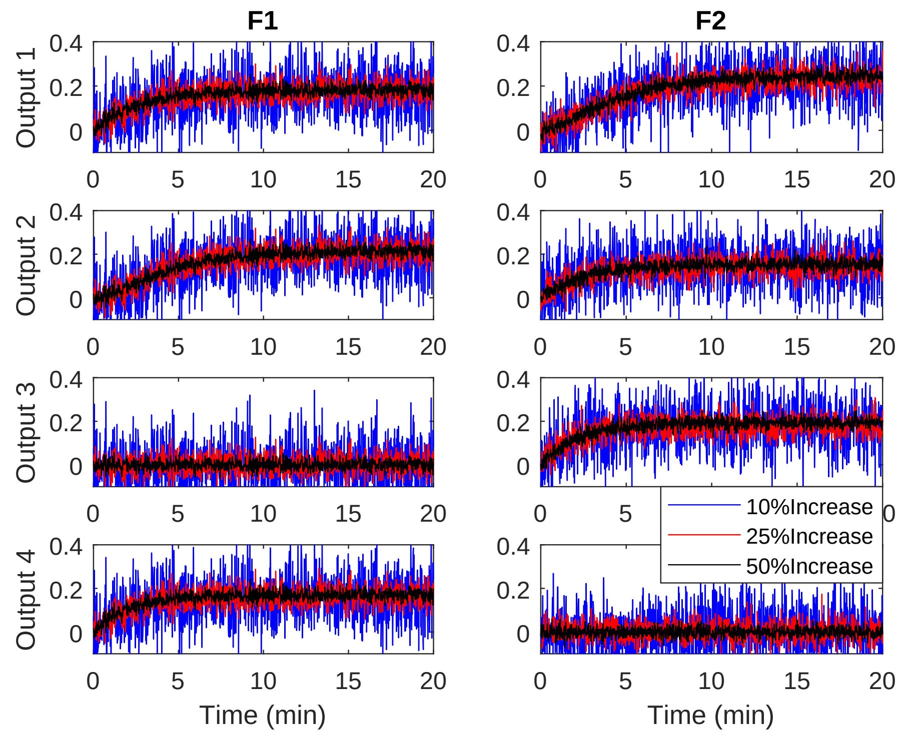



**(b)** Normalized step responses for stochastic non-linear model

**Figure 2.7:** Step responses and normalized step responses for stochastic non-linear model with 10%, 25% and 50% input increment and medium noise level (variance 2)




**(a)** Step responses for stochastic non-linear model




**(b)** Normalized step responses for stochastic non-linear model

**Figure 2.8:** Step responses and normalized step responses for stochastic non-linear model with 10%, 25% and 50% input increment and high noise level (variance 3)

model, the calculated step response of each input are fitted to their appropriate transfer functions.

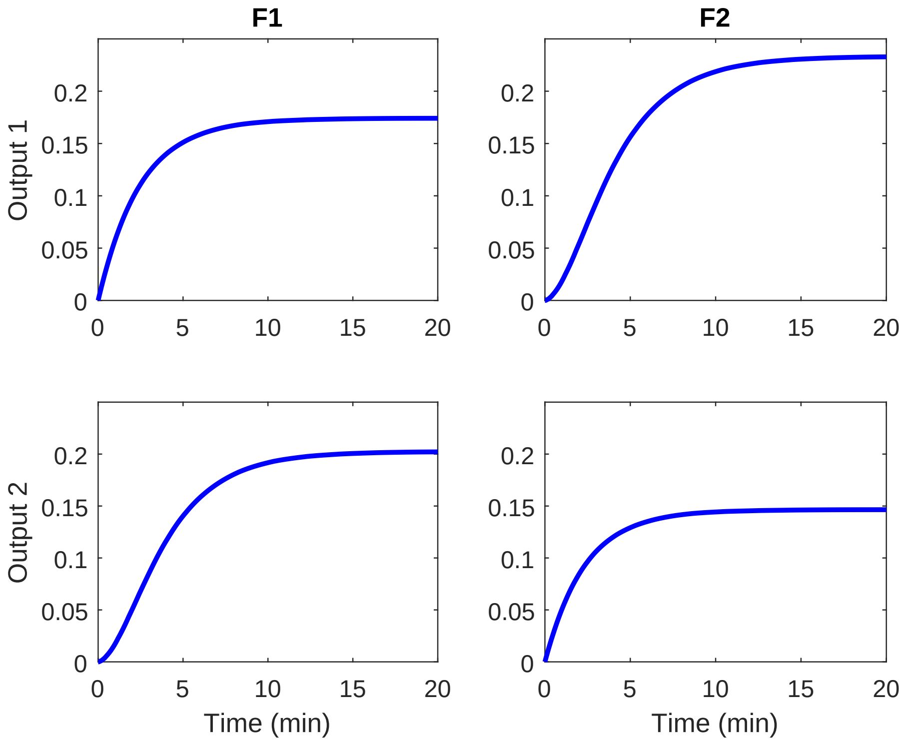



**Figure 2.9:** The normalized step responses of the estimated transfer function with 10% step increment of input

In order to carry out the transfer function identifications only 10% normalised step responses are considered. From Fig.2.9 only four step responses representing $h_1$ and $h_2$ with input $F_1$ and $F_2$ is shown since we are interested to measure the height and identify the transfer function only for Tank 1 and Tank 2. From estimation, the transfer function for $G_{11}$ and $G_{22}$ is estimated as first order whilst $G_{12}$ and $G_{22}$ is estimated as second order transfer function. These identified transfer functions as in equation (2.22) is represented in Table 2.10 below.

**Table 2.10:** The identified transfer function from the normalized step response

$$G_{11} = \frac{0.1740}{148s+1} \qquad G_{12} = \frac{0.2328}{(108s+1)(155s+1)}$$

$$G_{21} = \frac{0.2020}{(104s+1)(147s+1)} \qquad G_{22} = \frac{0.1465}{(139s+1)}$$

From the identified transfer functions we carried out an accuracy test by estimating the noise levels in the step responses. Then, to obtain a discrete-linear state space model and Markov parameters, we utilized a linear model predictive control toolbox provided to compute the discrete-linear model matrices and the Markov Parameters.

***Noise Estimation***

We estimate the measurement noise of the system by assuming that

$$Y = GU + E \tag{2.25a}$$
$$E = Y - GU \tag{2.25b}$$

where $E$ represents the unknown noise (measurement noise) with unknown mean and variance. From the normalised step responses $Y$, the noise is estimated using an equation as in equation (2.25b) involving the transfer function $G$ and compare the values with the measured noise included in section 2.3.4. The mean and variance of the noise with a normalized 10% step increment response are computed. The value of mean and variance estimation is from measured data, $h_1$ and $h_2$ for three levels of noise and with input $F_1$ and $F_2$. The estimations for mean and variance averaging on the different step levels are provided in Table 2.11 and by comparison, it shows that the values of variances identified from the transfer functions yield accurate estimation. As for the mean values are very close to 0, it can be concluded that the mean of the noise is approximated fairly accurate since the measurement noise for the three different noise level cases has a zero mean.

**Table 2.11:** Estimation of noise mean and variance averaging the step level from the identified transfer functions

| Average | Low | Medium | High |
|---|---|---|---|
| Mean | $3.5e^{-3}$ | 0.0184 | 0.0367 |
| Variance | 1.8747 | 4.8858 | 9.8635 |

***Markov Parameters***

Next is the analysis of Markov parameters, also known as impulse response coefficients, for a chosen sampling time. The discrete-time linear state space model is calculated and then the error of tolerance is set to $10^{-4}$, using Markov parameters the states of the discrete-time model can be determined. The discrete-time state space models are represented as:

$$x_{k+1} = A_d x_k + B_d u_k + E_d d_k \tag{2.26a}$$
$$y_k = C_d x_k + D_d u_k \tag{2.26b}$$

From section 2.2 it is known that four states are required to describe the system but this might not be the case in the discrete-time model. If the smallest singular value of the discrete state space realization is larger than the error tolerance meaning it has a higher number of states. The Hankel Singular values of the discrete-time state space model is as shown as in Fig.2.12 with the dimension of $A_d$ is four. In the next experiment, we compute the approximation of the steady-state model of the system and compare the step responses with the one that represents the actual transfer function from section 2.3.5. Referring to Fig.2.13a the comparisons of step responses between approximated

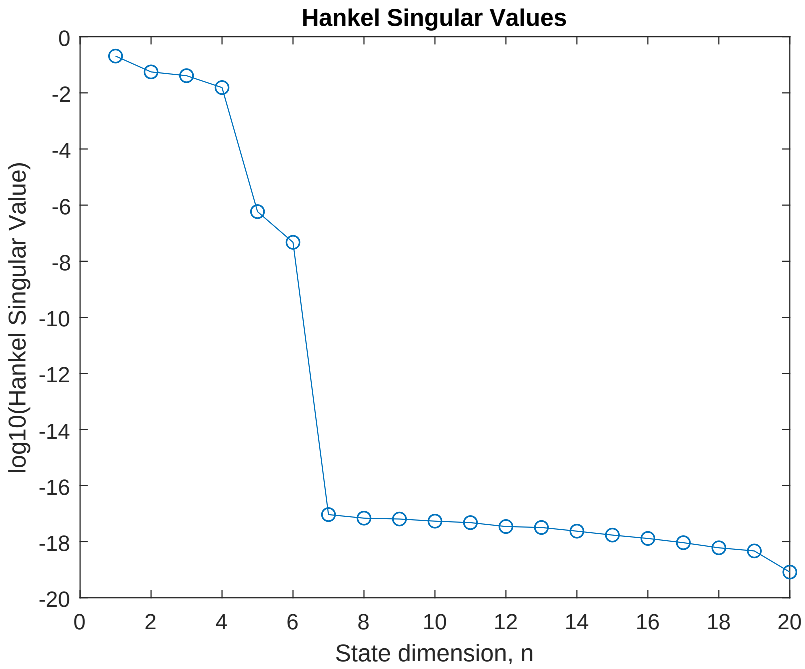


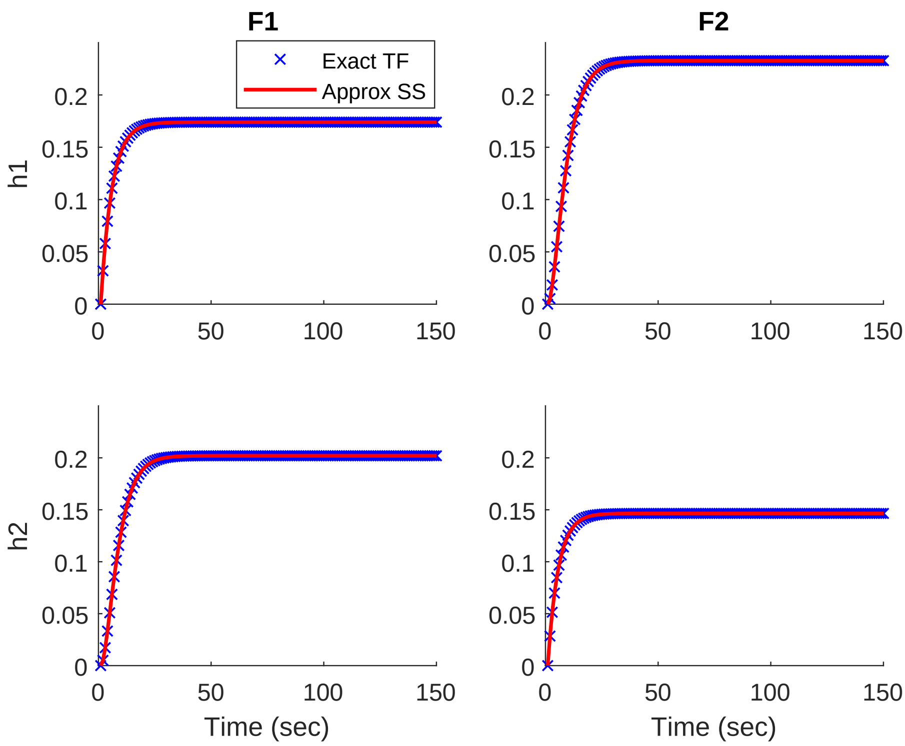

**Figure 2.12:** Hankel Singular Values with $N = 20$

steady-state model and model from the identified transfer function is presented. Blue dotted line is the step response for the identified transfer function and the red line is from the approximated state space model. Through observation, both responses are identical validating the approximated state space model.

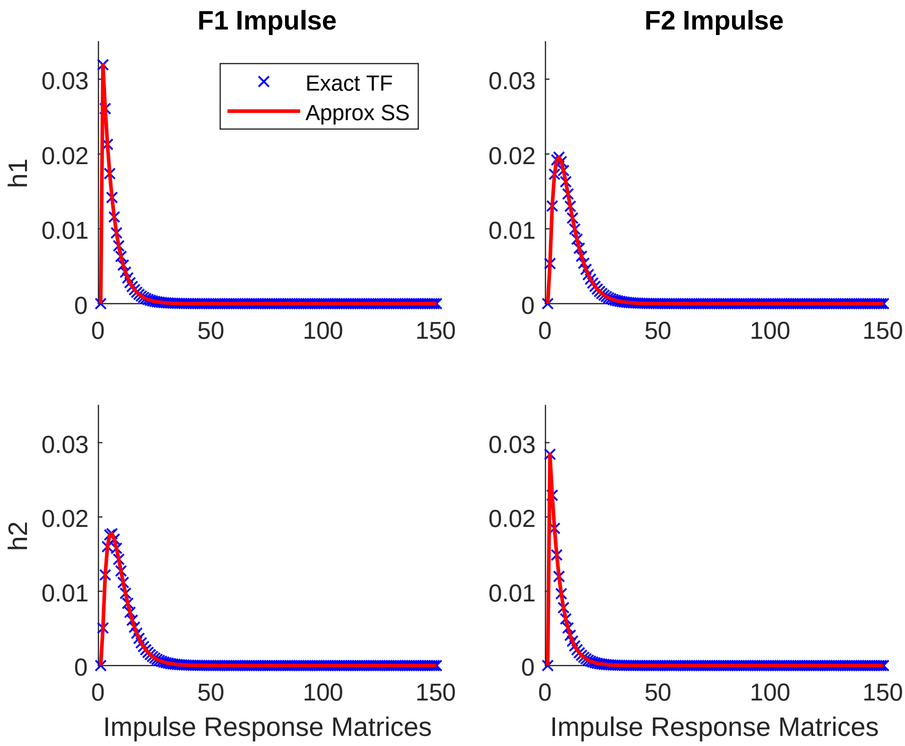
The impulse response of the system for both approximated steady state model identified transfer function is computed for Markov parameters reinforcement. The results of the responses can be perceived in Fig.2.13b where both of the responses shows an identical impulse response, likewise validating the approximated state space model.

## 2.4 Linear Discrete-time State Space Model

A dynamic model of a system can be described in a various way. From the derivation of the mathematical model, it is possible to obtain the underlying information of the system and implement a control algorithm to the system. Most of the systems or processes are usually described by a state space system and by investigating the state of a system at a certain time and its present and future inputs, it is possible to predict the output in the future [25]. State space models can be either non-linear or linear form and usually, a real system or process is described by a non-linear model whereas, in order to estimate and control the system, most mathematical tools are more accessible to a linear model. Therefore, in this section, we demonstrate the transformation of a non-linear continuous model of a modified quadruple tank system described as deterministic-stochastic differential equations into a linear discrete-time state space model. Several works have been done on similar four tanks system regarding the modelling of the dynamic of the system.




(a) Computation of step response of the system




(b) Computation of impulse response of the system

**Figure 2.13:** Step responses and normalized step responses for stochastic non-linear model with 10%, 25% and 50% input increment and high noise level (variance 3)

A full description of linearization of the model for the four tank system is presented in [23], [17] and in [26] the linearization is described in detail using the Jacobian matrix formation to represent the system in state space model.

### 2.4.1 Linear System Realization

Linearization is required to find the linear approximation to analyze the behaviour of the non-linear function, given a desired operating point. We apply the first-order term only of Taylor expansion by truncation around the steady state of the non-linear differential equations, $f(x(t), u(t), d(t))$ and consider the derivative of the state variable, $x$. This derivative is defined as a function, $f$,

$$f(x(t), u(t), d(t), p) = \begin{pmatrix} \rho\gamma_1 u_1(t) + \alpha_3 - \alpha_1 \\ \rho\gamma_2 u_2(t) + \alpha_4 - \alpha_2 \\ \rho(1-\gamma_2)u_2(t) + \rho d_1(t) - \alpha_3 \\ \rho(1-\gamma_1)u_1(t) + \rho d_2(t) - \alpha_4 \end{pmatrix} \tag{2.27}$$

where $\alpha_i$ is given by

$$\alpha_i = \rho a_i \sqrt{x_i(t)} \sqrt{\frac{2g}{\rho A_i}} \qquad i = 1, 2, 3, 4 \tag{2.28}$$

and $p$ denote the vector containing all the parameters of the system, for the full description of the parameter see Table 2.1. The Jacobian of $f$ with respect to the state-variables are

$$J_x(x(t), u(t), d(t), p) = \begin{pmatrix} -\beta_1 & 0 & \beta_3 & 0 \\ 0 & -\beta_2 & 0 & \beta_4 \\ 0 & 0 & -\beta_3 & 0 \\ 0 & 0 & 0 & -\beta_4 \end{pmatrix} \tag{2.29}$$

where $\beta_i$ is given by

$$\beta_i = \frac{1}{\sqrt{x_i(t)}} \sqrt{\frac{a_i^2 g \rho}{2A_i}} \qquad i = 1, 2, 3, 4 \tag{2.30}$$

Similarly the Jacobian of $f$ with respect to the manipulated variables giving

$$J_u(x(t), u(t), d(t), p) = \begin{pmatrix} \rho\gamma_1 & 0 \\ 0 & \rho\gamma_2 \\ 0 & \rho(1-\gamma_2) \\ \rho(1-\gamma_1) & 0 \end{pmatrix} \tag{2.31}$$

and lastly the Jacobian of $f$ with respect to the disturbance variables are

$$J_d(x(t), u(t), d(t), p) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \rho & 0 \\ 0 & \rho \end{pmatrix} \tag{2.32}$$

In this case, we introduced the deviation variables as

$$X(t) = x(t) - x_s \quad U(t) = u(t) - u_s \quad D(t) = d(t) - d_s \tag{2.33}$$

and defined the Jacobian matrices evaluated around a stationary point $x_s, u_s, d_s$ to be

$$A_c = J_x(x_s, u_s, d_s, p) \quad B_c = J_u(x_s, u_s, d_s, p) \quad E_c = J_d(x_s, u_s, d_s, p) \tag{2.34}$$

With these matrices the first order Taylor approximation around the steady state point are given as

$$\begin{aligned} f(x(t), u(t), d(t), p) &\approx f(x_s, u_s, d_s, p) + A_c X(t) \\ &\quad + B_c X(t) + E_c D(t) \\ &= A_c X(t) + B_c X(t) + E_c D(t) \end{aligned} \tag{2.35}$$

For the measurement and controlled variables we introduced $Y(t)$ and $Z(t)$ respectively and the linearized system of the modified quadruple tank system as

$$\dot{X}(t) = A_c X(t) + B_c X(t) + E_c D(t) \qquad X(t_0) = 0 \tag{2.36a}$$

$$Y(t) = C X(t) \tag{2.36b}$$

$$Z(t) = C_z X(t) \tag{2.36c}$$

where the $C$ matrices are defined as

$$C = C_z = \begin{pmatrix} \frac{1}{\rho A_1} & 0 & 0 & 0 \\ 0 & \frac{1}{\rho A_2} & 0 & 0 \end{pmatrix} \tag{2.37}$$

### 2.4.2 Discretization of a Linear System

The dynamics of the modified quadruple tank system is now described as (2.36) and to use this linear continuous model of the system to be subjected to MPC, the model needs to be discretized by assuming zero-order-hold (ZOH) of the variables at specified sampling points, that is assuming the exogenous variables are constant between sampling points. The aim is to have a linear discrete-time state space model with piecewise constant $u_k$, $d_k$ in a form of

$$x_{k+1} = A_d x_k + B_d u_k + E_d d_k \tag{2.38a}$$

$$y_k = C_d x_k + D_d u_k \tag{2.38b}$$

with discrete-time consideration

$$\begin{aligned} t_k &= t_0 + kTs, \qquad k = 0, 1, 2... \\ x_k &= x(t_k) \end{aligned}$$

and assuming the inputs on the ZOH is

$$u(t) = u_k, \qquad t_k \le t \le t_{k+1}$$

then the solution of (2.38) with respect to $u$ is given as

$$x_{k+1} = x(t_{k+1}) \tag{2.39a}$$

$$= e^{A(t_{k+1} - t_k)} x_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1} - \tau)} B u(\tau) d\tau \tag{2.39b}$$

$$= \left[ e^{AT_s} \right] x_k + \left[ \int_0^{T_s} e^{A\eta} B d\eta \right] u_k \tag{2.39c}$$

By comparing both equations (2.36) and (2.39) and similar result can be obtained for disturbances variable $d(t)$ giving

$$\begin{array}{lll} A_d = e^{AT_s} & B_d = \int_0^{T_s} e^{A\tau} B d\tau & E_d = \int_0^{T_s} e^{A\tau} E d\tau \\ C_d = C & D_d = D & \end{array} \tag{2.40}$$

where $A_d, B_d, E_d$ can be computed with

$$\begin{aligned} \begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} &= \exp\left( \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} T_s \right) \\ \begin{bmatrix} A_d & E_d \\ 0 & I \end{bmatrix} &= \exp\left( \begin{bmatrix} A & E \\ 0 & I \end{bmatrix} T_s \right) \end{aligned} \tag{2.41}$$

For this particular work, the continuous state space representation matrices were discretized with $T_s = 30s$ assuming ZOH.

Considering the stochastic part of the model, a piecewise constant process noise, $w$ measurement noise, $v$ and uncertainty of the initial state, $x_0$ to the process is added. The linear discrete model from (2.38) is expanded into a stochastic version as in the equation below

$$x_{k+1} = A_d x_k + B_d u_k + E_d(d_k + w_k) \tag{2.42a}$$
$$y_k = C_d x_k + v_k \tag{2.42b}$$
$$z_k = C_{dz} x_k + v_k \tag{2.42c}$$

subject to

$$x_0 \sim N(\bar{x_0}, P_p), \quad w_k \sim N(0, Q), \quad v_k \sim N(0, R) \tag{2.43}$$

where $Q, R$ is given by

$$Q = \begin{bmatrix} 12.5^2 & 0 \\ 0 & 12.5^2 \end{bmatrix} \quad R = \begin{bmatrix} 2^2 & 0 & 0 & 0 \\ 0 & 2^2 & 0 & 0 \end{bmatrix}$$

and $P_p$ is given by

$$P_p = \begin{bmatrix} 0.1^2 & 0 & 0 & 0 \\ 0 & 0.1^2 & 0 & 0 \\ 0 & 0 & 0.1^2 & 0 \\ 0 & 0 & 0 & 0.1^2 \end{bmatrix}$$

### 2.4.3 Linear Discrete-time State Space Representation

In order to rewrite the difference equation system representation (2.38) in a more structured form, the Markov parameters are introduced. It is a discrete impulse coefficient of a discrete state space model. The Markov parameters are calculated to avoid making iterative simulations to keep only the matrix-vector multiplications. In doing so, a significant time saving is introduced to the control algorithm and to have an observer canonical form with minimal realization. Let $H_i$ denote the Markov parameters at the

$i'th$ sampling time after an unit-impulse, then to obtain the Markov parameters from $u$ to $y$ is given as

$$H_i = \begin{cases} 0 & i = 0 \\ C{A_d}^{i-1}B & i = 1, 2, ...N \end{cases} \tag{2.44}$$

$N$ is assigned value to be sufficiently large so that the impulse response can reach the steady state. The Markov parameters for $u$ to $z$, $d$ to $y$ and $d$ to $z$ is computed the same way and by replacing the appropriate matrices accordingly. With all the information being gathered, it can be rewritten in a matrix form of

$$Y = \Phi x_0 + \Gamma U \tag{2.45}$$

where $Y$, $\Phi$, $U$ are

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_i \end{bmatrix} \quad \Phi = \begin{bmatrix} CA_d \\ C{A_d}^2 \\ C{A_d}^3 \\ \vdots \\ C{A_d}^i \end{bmatrix} \quad U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_i \end{bmatrix}$$

while $\Gamma$ is obtained from the calculated Markov parameters, $H_i$, $i = 1, 2, ...N$

$$\Gamma = \begin{bmatrix} H_1 & 0 & 0 & \dots & 0 \\ H_2 & H_1 & 0 & \dots & 0 \\ H_3 & H_2 & H_1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ H_N & H_{N-1} & H_{N-2} & \vdots & H_1 \end{bmatrix}$$

As for the system with disturbances, the state space model can be represented as

$$Y = \Phi x_0 + \Gamma_u U + \Gamma_d D \tag{2.46}$$

where

$$D = \begin{bmatrix} d_1 & d_2 & d_3 & \dots & d_i \end{bmatrix}^T$$

From equations (2.45) and (2.46), $\Phi$ and $\Gamma$ can be used for the prediction part from the Kalman filter for the model predictive control strategy in the next chapter.

## 2.5 Operating Window in Steady State

In this section, the operating window of the MQT system is featured. We elongate the study and analysis around the steady state to acquire its comportment and to develop the operating window which is sets of set points selection boundaries. These operating windows give some basic ideas and guideline on choosing the appropriate set points in certain specific conditions of operations.

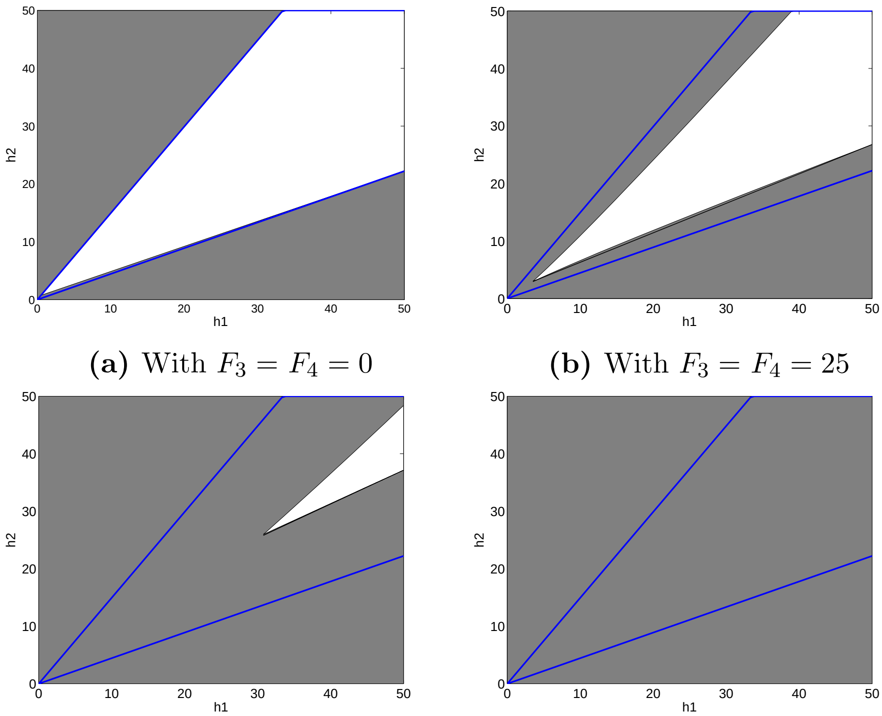
At steady state, the outflows and inflows of each tank are identical. Therefore, (2.4) in combination with (2.13) can be combined into

$$q_{1,s} = q_{1,in,s} + q_{3,s} = \gamma_1 F_{1,s} + (1 - \gamma_2)F_{2,s} + F_{3,s} \tag{2.47a}$$

$$q_{2,s} = q_{2,in,s} + q_{4,s} = \gamma_2 F_{2,s} + (1 - \gamma_1)F_{1,s} + F_{4,s} \tag{2.47b}$$

$$q_{3,s} = q_{3,in,s} + F_{3,s} = (1 - \gamma_2)F_{2,s} + F_{3,s} \tag{2.47c}$$

$$q_{4,s} = q_{4,in,s} + F_{4,s} = (1 - \gamma_1)F_{1,s} + F_{4,s} \tag{2.47d}$$




**(a)** With $F_3 = F_4 = 0$

**(b)** With $F_3 = F_4 = 25$

**(c)** With $F_3 = F_4 = 75$

**(d)** With $F_3 = F_4 = 100$

**Figure 2.15:** Operating window with $\gamma_i$ in the RHP

$$m_{1,s} = \left(\frac{\gamma_1 F_{1,s} + (1-\gamma_2)F_{2,s} + F_{3,s}}{\alpha_1}\right)^2 \tag{2.48a}$$

$$m_{2,s} = \left(\frac{\gamma_2 F_{2,s} + (1-\gamma_1)F_{1,s} + F_{4,s}}{\alpha_2}\right)^2 \tag{2.48b}$$

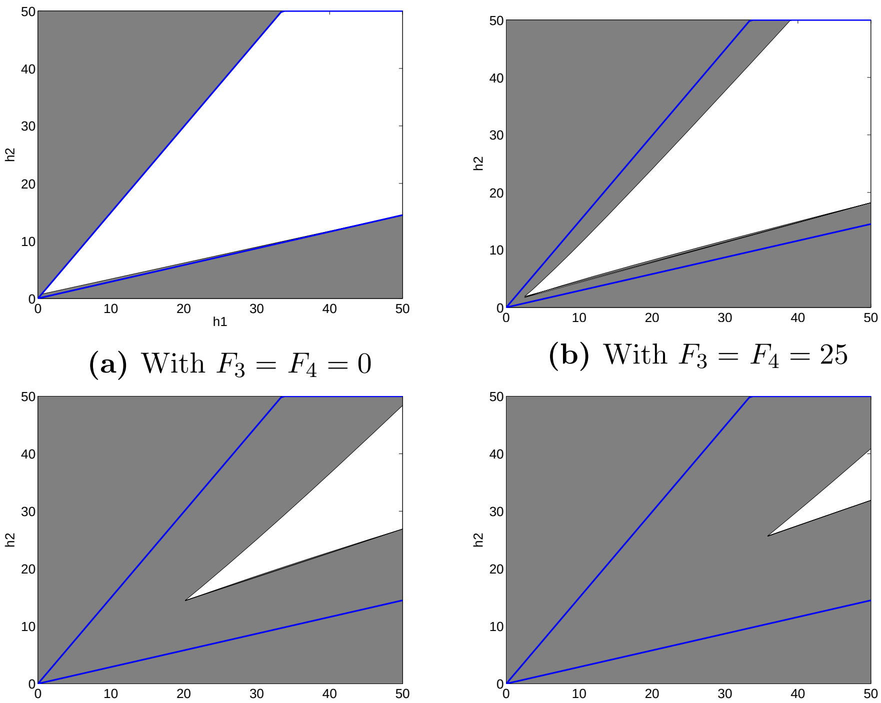
$$m_{3,s} = \left(\frac{(1-\gamma_2)F_{2,s} + F_{3,s}}{\alpha_3}\right)^2 \tag{2.48c}$$

$$m_{4,s} = \left(\frac{(1-\gamma_1)F_{1,s} + F_{4,s}}{\alpha_4}\right)^2 \tag{2.48d}$$

$$h_{i,s} = \beta_i m_{i,s} \qquad i = 1,2,3,4 \tag{2.49}$$

If the liquid heights in Tank 1 and Tank 2, $h_{1,s}$ and $h_{2,s}$, are specified, the corresponding masses, $m_{1,s} = h_{1s}/\beta_1$ and $m_{2s} = h_{2s}/\beta_2$, would also be specified and so would the outflow rates from Tank 1 and Tank 2, $q_{1,s} = \alpha_1\sqrt{m_{1,s}}$ and $q_{2,s} = \alpha_2\sqrt{m_{2,s}}$. Consequently, the required steady state manipulable flow rates, $F_{1,s}$ and $F_{2,s}$, must satisfy

$$\underbrace{\begin{bmatrix} \gamma_1 & 1-\gamma_2 \\ 1-\gamma_1 & \gamma_2 \end{bmatrix}}_{=M} \begin{bmatrix} F_{1,s} \\ F_{2,s} \end{bmatrix} = \begin{bmatrix} q_{1,s} - F_{3,s} \\ q_{2,s} - F_{4,s} \end{bmatrix} \tag{2.50}$$




**(a)** With $F_3 = F_4 = 0$

**(b)** With $F_3 = F_4 = 25$

**(c)** With $F_3 = F_4 = 75$

**(d)** With $F_3 = F_4 = 100$

**Figure 2.16:** Operating window with $\gamma_i$ in the LHP

The coefficient matrix, $M$, is singular when

$$\det M = \gamma_1\gamma_2 - (1-\gamma_1)(1-\gamma_2) = 0 \tag{2.51}$$

i.e when

$$\gamma_2 = 1 - \gamma_1 \tag{2.52}$$

and (2.50) does not have a unique solution. When $\det M \neq 0$, (2.50) has a unique solution given by

$$\begin{bmatrix} F_{1,s} \\ F_{2,s} \end{bmatrix} = \begin{bmatrix} \gamma_1 & 1-\gamma_2 \\ 1-\gamma_1 & \gamma_2 \end{bmatrix}^{-1} \begin{bmatrix} q_{1,s} - F_{3,s} \\ q_{2,s} - F_{4,s} \end{bmatrix} \tag{2.53}$$

For the case when $\det M = 0$, an extra condition, e.g. $F_{1,s} = F_{2,s}$, can be used to obtain a unique solution. In the case $F_{1,s} = F_{2,s}$, the unique solution is given by

$$F_{1,s} = F_{2,s} = \frac{q_{1,s} - F_{3,s}}{\gamma_1 + (1-\gamma_2)} = \frac{q_{2,s} - F_{4,s}}{(1-\gamma_1) + \gamma_2} \tag{2.54}$$

if it exists. The solution only exists if the latter equality in (2.54) is satisfied. To solve (2.53), temporarily let $b_1 = (q_{1,s} - F_{3,s})$ and $b_2 = q_{2,s} - F_{4,s}$

$$\begin{bmatrix} F_{1,s} \\ F_{2,s} \end{bmatrix} = \frac{1}{detM} \begin{bmatrix} \gamma_2 b_1 - (1-\gamma_2) b_2 \\ \gamma_1 b_2 - (1-\gamma_1) b_1 \end{bmatrix} \tag{2.55}$$

With the manipulation of (2.55) the equation can be transform into two condition,

***Condition 1: det M > 0***

$$\frac{1-\gamma_1}{\gamma_1} \leq \frac{q_{2,s} - F_{4,s}}{q_{1,s} - F_{4,s}} \leq \frac{\gamma_2}{1-\gamma_2} \tag{2.56}$$

***Condition 2: det M < 0***

$$\frac{\gamma_2}{1-\gamma_2} \leq \frac{q_{2,s} - F_{4,s}}{q_{1,s} - F_{4,s}} \leq \frac{1-\gamma_1}{\gamma_1} \tag{2.57}$$

By inserting (2.49) into (2.13) and $\eta_i$ is given by,

$$\eta_i = \frac{\alpha_i}{\sqrt{\beta_i}} \qquad i = 1, 2 \tag{2.58}$$

the possible feasible region of set points of the heights in Tank 2, $h_{2,s}$ can de determined by

$$h_{2,lb} \leq h_{2,s} \leq h_{2,ub} \tag{2.59}$$

where $h_{2,lb}$ and $h_{2,up}$ is upper and lower bound of $h_2$ respectively with

$$h_{2,lb} = \left[\frac{\left(\frac{1-\gamma_1}{\gamma_1}\right)\left(\eta_1\sqrt{h}_{1,s} - F_{3,s}\right) + F_{4,s}}{\eta_2}\right]^2$$

$$h_{2,ub} = \left[\frac{\left(\frac{\gamma_2}{1-\gamma_2}\right)\left(\eta_1\sqrt{h}_{1,s} - F_{3,s}\right) + F_{4,s}}{\eta_2}\right]^2$$

This corresponds to solving

$$f(x_s, u_s, d_s, p) = 0 \tag{2.60a}$$
$$y_s = g(x_s, p) \tag{2.60b}$$
$$z_s = h(x_s) \tag{2.60c}$$

for $x_s$ when $u_s$, $d_s$ and $p$ are given.

We derived and investigated the steady state to determine the possible feasible region of set points and a suitable operating window for the modified quadruple tank system. It is non trivial decision to choose the appropriate range of set point, $h_2$ for certain conditions such as the value of $\gamma_i$ and disturbances $F_3$ and $F_4$ with min and max bound of disturbances selections is $0 \leq d \leq 100$. The simulation of the analysis shows the possible region of setpoint selections.

The white region in Fig.2.15 and Fig.2.16 shows the operating window that is feasible for different disturbance conditions. Referring to Fig.2.14a and Fig.2.16a, although the feasible region to choose $h_2$ is wide, it can be seen that it is bounded with upper and lower limits, and if we refer to Fig.2.14c and Fig.2.16c it is almost not possible to choose the middle set points. Comparing Fig.2.14d and Fig.2.16d, with a different selection of the fixed fraction of the water flow $\gamma_i$, the modified quadruple tank system is able to operate at the maximum value of disturbances during LHP operation.

## 2.6 Summary

The first essential part of the research work which is the comprehension of the MQT system is presented in this chapter by extracting the governing equations of the system and expressing it in a structured mathematical model through simulation and analysis.

- The presented model which is provided in Paper A describes the modified quadruple tank system as a deterministic and stochastic ordinary differential equations (ODEs) model. In particular, it was presented how step tests can be conducted and used to identify the transfer function from the normalized step response.
- The steady state analysis is investigated in order to develop an operating window to determine the selection of set points since not all set points combinations are possible for the system to attain its desired outcomes. This operating window for the MQT system is also presented in Paper A, it is not negligible and useful for future controller design.
- The chapter has also illustrated how linear models of MQT processes can be obtained by linearization and discretization, and then we reconstruct the structure of the model to make it more presentable by using Markov parameters so to have a model that is suitable for computer control analysis, as exhibited in Paper B.

CHAPTER 3

# State Estimation

This chapter focuses on the state estimation for the estimator part of the MPC by incorporating a Kalman filter. The Kalman filter consists of two parts, the filtering part and the predictions part. The filtered part is utilized to estimates the current state based on the model and the measurements whilst the prediction part is used by the constrained regulator to predict the future output trajectory, given an input trajectory. In this chapter, a brief introduction of state estimation is presented and in the next section description of the Kalman filter is provided. The Static Kalman filter algorithm derived from the Discrete-time Kalman filter is considered in this study. Finally, in assuring to have an offset-free control, a disturbance model is included in the model of the system and the formulation is presented in the final section.

## 3.1 Discrete-time Linear System for State Estimation

The block diagram in Fig.3.1 shows that the MPC implementation segment consists of an estimator and an optimizer. The state estimation is the core of the MPC since the estimator incorporates feedback into the MPC and provides estimation to the optimizer or regulator to proceed in the next step for predictions.

In order to predict the future dynamic behaviour of the MQT system, it is important to have an estimation of the current states of the process since it is unknown and cannot be measured directly. Therefore the state of the process, $x_k$ need to be estimated and this can be done by the measurement of the process which is somehow related to the state.

To this end, the discrete-time linear stochastic state space model derived in section 2.4 in the form of

$$x_{k+1} = A_d x_k + B_d u_k + E_d d_k + E d w_k \tag{3.1a}$$
$$y_k = C_d x_k + v_k \tag{3.1b}$$

subject to

$$w_k \sim N(0, Q), v_k \sim (0, R) \tag{3.2}$$

where the process noise $w_k$ and measurement noise $v_k$ are distributed as

$$\begin{bmatrix} w_k \\ v_k \end{bmatrix} \sim N_{iid}\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \right) \tag{3.3}$$

where $R$ and $Q$ is the covariance matrix of measurement error, $w_k$ and disturbances variable, $v_k$ accordingly, $S$ is the covariance matrix between disturbance variable, $v_k$ and

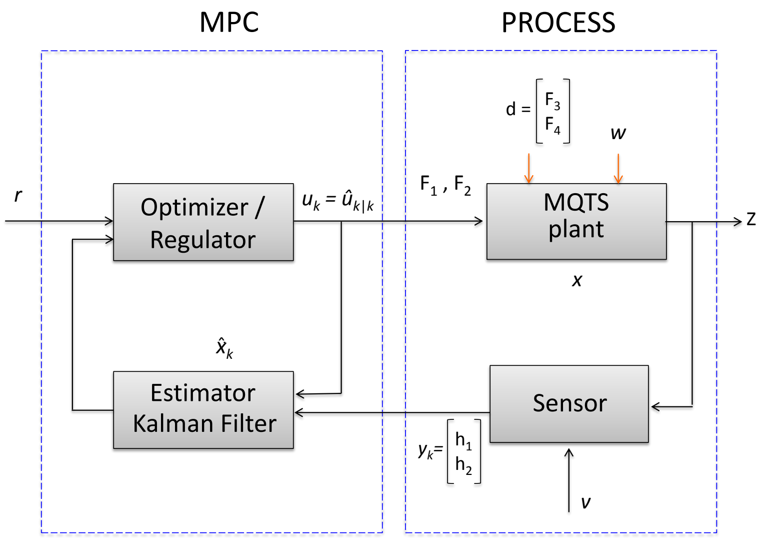



**Figure 3.1:** Block diagram of the control structure for the MQT system process

measurement error, $w_k$. The covariance matrices Q, R, and the cross-covariance matrix S are positive definite and symmetric. The distribution of the initial state is given by

$$x_{0|-1} \sim N(\hat{x}_{0|-1}, P_{0|-1}) \tag{3.4}$$

and is independent of process and measurement noise.

## 3.2 Kalman Filter

Given the measurement is $y_k$ with $k = 0, 1 \ldots N$, the discrete-time interval $[0, N]$ and let $Y_N$ represents the set of measurements from discrete-time $k = 0$ to $k = N$

$$Y_N = y_0, y_1, \ldots, y_N \tag{3.5}$$

The main aspect in the state estimation problem is to obtain an estimation of state, $\hat{x}_{k|k}$ from the current state, $x_k$, based on the measurement data, $y_k$ at time $k$ and in order to obtain this state estimation, Kalman filter is used. It is a recursive approach based on R.E Kalman [27] where the main role of Kalman filter is to minimize the sum of squared errors between the current state, $x_k$ and the state estimation, $\hat{x}_{k|k}$. For optimum filtration, the estimation model is assumed identical to the real system, both process and measurement noise is assumed white and the source of the covariances of the noise is assumed precisely known.

There are two sets of recursions of Kalman filter for the discrete-time stochastic model, first is the filter part (a data update) where the filtered variables is based on the current measurement, $(k|k)$ and the other one is a step prediction part (a time update) where the predicted variable depends on the previous measurement, $(k|k-1)$.

Given the data set $Y_k$, the conditional expectation of the state, $x_k$ and the process noise, $w_k$ is computed in relates to the filtered part.

$$\hat{x}_{k|k} = E\left\{\hat{x}_k|Y_k\right\} \tag{3.6a}$$
$$\hat{w}_{k|k} = E\left\{w_k|Y_k\right\} \tag{3.6b}$$

associated covariances of the state estimate is

$$P_{k|k} = Var\left\{\hat{x}_k|Y_k\right\} \tag{3.6c}$$

Given the data set $Y_k$, the conditional expectation of the state, $x_k$ is computed in relates to the prediction part.

$$\hat{x}_{k+1|k} = E\left\{\hat{x}_{k+1|k}|Y_k\right\} \tag{3.7a}$$
$$P_{k+1|k} = Var\left\{\hat{x}_{k+1|k}|Y_k\right\} \tag{3.7b}$$

# 3.3 Discrete-time Kalman Filter

In this section, the details of discrete-time Kalman filter implementation is presented. In the algorithm, unknown variables are estimated from the MQT system which is defined with disturbance and measurement noise. Assuming at stationary point $t = t_k$ and the measurement $y_k = y(t_k)$, the filtering part can be performed by calculating

$$\hat{y}_{k|k-1} = C\hat{x}_{k|k-1} \tag{3.8a}$$
$$e_k = y_k - \hat{y}_{k|k-1} \tag{3.8b}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx,k}e_k \tag{3.8c}$$
$$\hat{w}_{k|k} = K_{fw}e_k \tag{3.8d}$$
$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k + \hat{w}_{k|k} \tag{3.8e}$$

by using the coefficients

$$R_{e,k} = CP_{k|k-1}C^T + R \tag{3.9a}$$
$$K_{fx,k} = P_{k|k-1}C^T R_{e,k}^{-1} \tag{3.9b}$$
$$K_{fw} = SR_{e,k}^{-1} \tag{3.9c}$$

and the following expression for one-step prediction can be achieved

$$P_{k+1|k} = AP_{k|k}A^T + Q_{k|k} - AK_{fx,k}S^T - SK_{fx,k}^T A^T \tag{3.10}$$

***Computations of the Kalman filter***

With the initial condition given by equation (3.4) and at sample $k = 0$, the recursive computation of the static Kalman filter starts. New measurement of the system, $y_k$, is given to the state estimator for every sample of $k = 0, 1, 2...$ where $e_k$ is the error between the actual measurement, $y_k$ and the predicted measurement, $\hat{y}_{k|k-1}$ is computed, as shown in equation (3.8b). From equation (3.8a) the predicted measurement is predicted from the actual measurement, $y_k$ at sample $k$ but computed at sample $k-1$ provided by a set of data from $y_{k-1}$ and based on the one-step prediction as in equation (3.8c). The one-step prediction, $\hat{x}_{k|k-1}$ is the prediction from the current state, $x_k$ at sample $k$ which is computed at sample $k-1$. The filtered state estimation, equation (3.8c) is computed using $e_k$ and $\hat{x}_{k|k-1}$ and the filtered process noise, equation (3.8d)is estimated by using $e_k$. Finally the one-step prediction given by equation (3.8e) is computed for the Kalman filter to prepare for the next process measurements.

## 3.4 Static Kalman Filter

The static Kalman filter follows the same structure as the discrete-time Kalman filter but for static Kalman filter, the covariance matrix $P_{k|k-1}$ is kept constant. Moreover, it is taken as in the limit $k \to \infty$ and with this assumption, the computation can be performed as Riccati equation. From equation (3.10) it can be rewritten into a difference equation form as

$$P_{k+1|k} = AP_{k|k-1}A^T + Q - (AP_{k|k-1}C^T + S)(CP_{k|k-1}C^T + R)^{-1}(AP_{k|k-1}C^T + S)^T \tag{3.11}$$

$P$ signifies the stationary one-step ahead state error covariance matrix obtained from the Discrete-time Algebraic Riccati Equation (DARE), yielding the following expression

$$P = APA^T + Q - (APC^T + S)(CPC^T + R)^{-1}(APC^T + S)^T \tag{3.12}$$

and in the stationary condition, the coefficients in equations (3.9) can be simplified as

$$R_e = CPC^T + R \tag{3.13a}$$

$$K_{fx} = PC^T R_e^{-1} \tag{3.13b}$$

$$K_{fw} = SR_e^{-1} \tag{3.13c}$$

and these coefficients is computed offline.

Since by using this limit as an approximation to the one-step matrix and the Kalman gains, $K_{fx}$ and $K_{fw}$ becomes constant matrices, it will lighten the complex computations of estimating the current state, $x_k$. From equations 3.8, the filtration and estimation updates for the static Kalman filter are computed. By solving the Riccati equation in equation (3.12), the stationary covariance matrix $P$ is obtained and the computation depends on the noise covariance matrices $Q$, $R$ and $S$.

### 3.4.1 Simulation for the Static Kalman Filter

The simulation for the static Kalman filter started with equation (3.1) with given inputs $u_k$ and $d_k$ at some random initial condition $x_0$ and constructed forward in time states, measurements and outputs. The stochastic variables were simulated beforehand with an appropriate mean and covariance. The command **randn** in Matlab was used to produce the stochastic realizations by using Cholesky factorization and a seed for **randn** was set to always have the same realization of noises for the simulations.

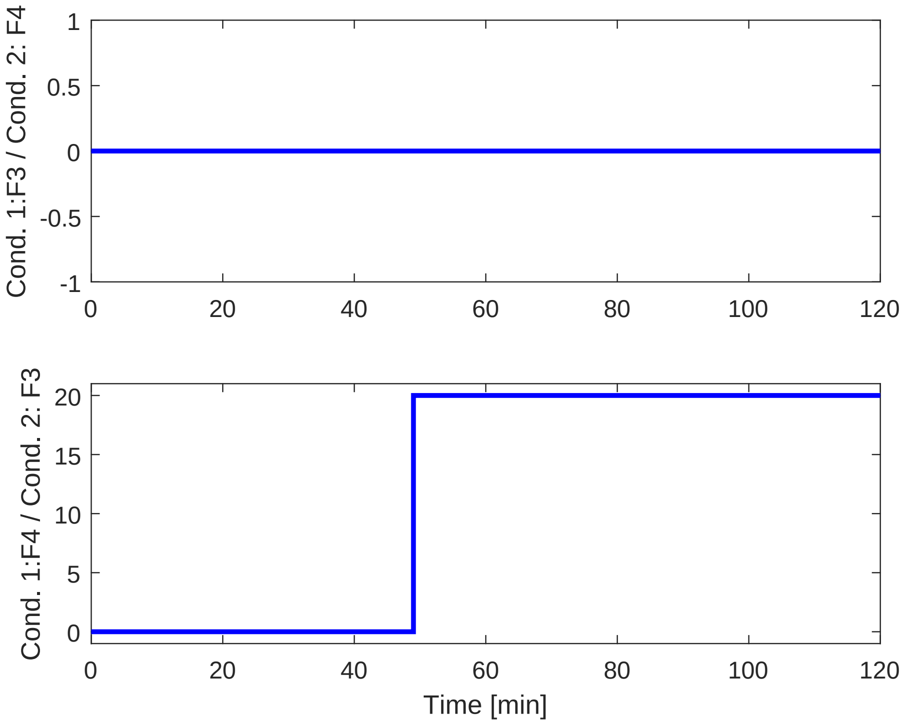



**Figure 3.2:** Disturbance Step Input $F_3$ and $F_4$ for **Condition 1** and **Condition 2** respectively

For testing purposes, the step change in the disturbances of $F_3$ and $F_4$ is applied at $t = 50min$, realized as 20 units of deviation from the steady state, as shown in Fig.3.2 individually for **Condition 1** and **Condition 2** respectively.

The simulation was done for 120 minutes with 30 seconds of sampling time. The simulation results for **Condition 1** and **Condition 2** are presented in Fig.3.3 and Fig.3.4 respectively. The role of the static Kalman filter is observed by comparing the measurements responses. The blue lines (Unfiltered) represents the actual measurements and the red lines (Filtered) represents the corresponding predicted measurements. It can be observed that the static Kalman filter performed satisfactorily where it estimated a prediction compatible with the simulated data with the effect of the step change in the disturbance variables, $d = [F_3 \quad F_4]^T$. This shows that with a step change included, the algorithm has the ability to predict both simulated measurements and outputs.

Generally, it can be clearly seen that the filter is well performed tracking the output trajectory from the noisy measurements and also it can cope well dealing with an impact of the unknown disturbance step. Throughout the following chapters, the static Kalman filter will be incorporated in the MPC design and implementation for the MQT system since the covariance matrix $P_{k|k-1}$ is kept constant and the computation of the controller is lightened due to the fact that covariance $P_{k+1|k}$ converges to equilibrium rather fast.

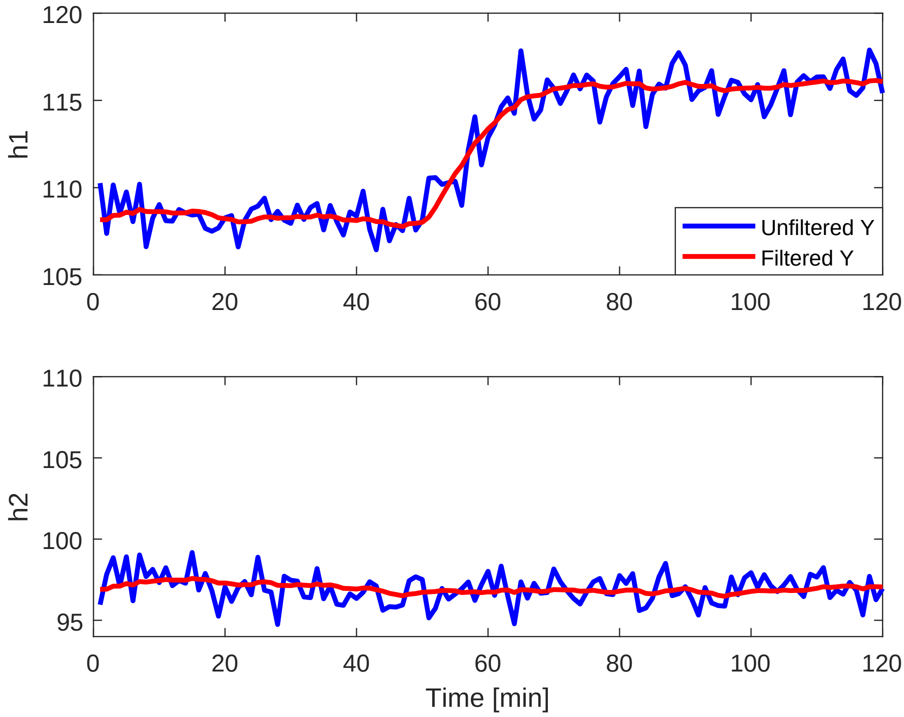



**Figure 3.3:** Kalman filter with measurement variable $h_1$ and $h_2$ with input disturbance of $F_3$

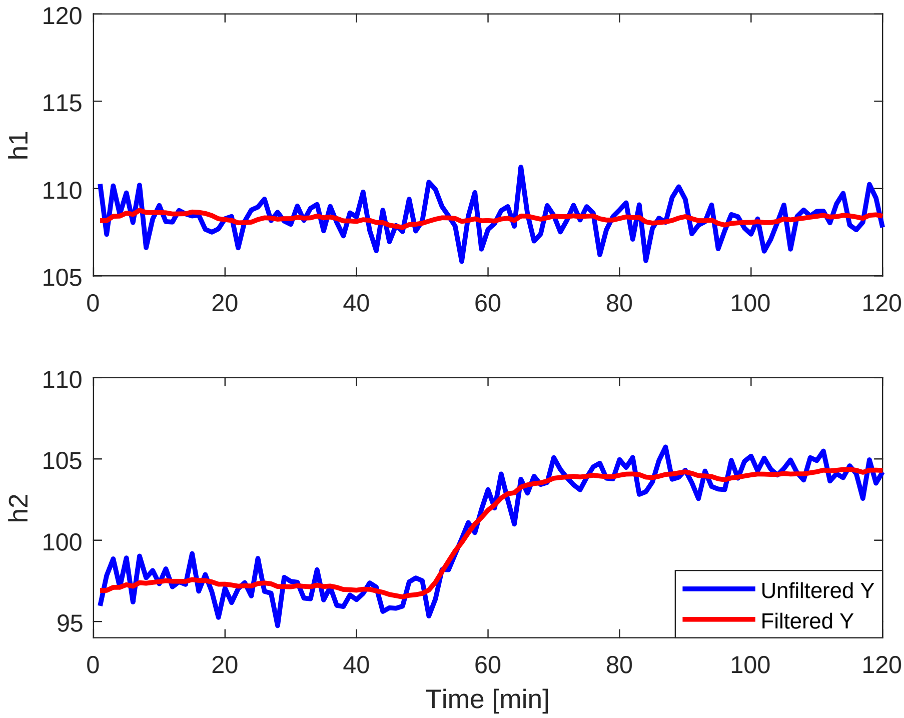



**Figure 3.4:** Kalman filter with measurement variable $h_1$ and $h_2$ with input disturbance of $F_4$

***Predictions from the Static Kalman Filter***

For future output prediction, the filtered state estimation, $\hat{x}_{k|k}$ is applied as the initial value and the filtered process-noise is included in the one-step prediction as given by equation (3.8). As for the predictions of the entire horizon, $N$ the process noise estimate $w_{k+j} = 0$ for $j > 0$ since $w_k$ and $v_k$ are correlated only for the current $k$. Thus, the state prediction for the next step, $(j+1)$ can be computed by

$$\hat{x}_{k+1+j|k} = A\hat{x}_{k+j|k} + Bu_{k+j|k}, \qquad j = 1, 2, ... \tag{3.14}$$

# 3.5 Offset-free Control

Uncertainties of parameters and hardware accuracy or limitations could lead to model mismatch and unmeasured disturbances, consequently instigating an offset. Due to this offset, the performance of the control system can be affected. Since model predictive control is a model based controller which the performance is fully dependent on the dynamic model of the system, both unmeasured disturbances and model mismatch should be taken into account to intensify the robustness of the controller.

For simulation purposes, we can choose any inputs for the plant to produce the desired output but at the same time, it has to be realistic. In closed-loop simulations, the inputs are computed for the next iterations to solve the optimization problem and to simulate the output responses of the system but the values computed by the controller cannot be simply implemented due to the accuracy and limitation issues. This could be considered as additional unmeasured disturbances in the existing disturbance variables and model mismatch.

Therefore, for practical purposes and to resolve this issue, an *offset-free control* concepts is applied where an augmented system from the original system is used. The system can be modified in many ways depending on the augmentation of the states. One way is to add the integration of the tracking error to the model of the system or to include a velocity form of the state space model or to modifies the model of the system by adding input or/and output disturbance model. Each and every approach has its own advantages and drawbacks [23].

In this thesis, the latter approach is used to achieve an offset-free control but only the input disturbance is included. This approach is considered due to the fact that by abolishing the output disturbance model, it would reduce the computational time of solving the equations of the system substantially [23].

## 3.5.1 Input Disturbance Model

A new stochastic model is formulated by introducing an input disturbance variable, $\eta_k$ as a separate state variable of the model which affects input, $u$ in the form of $U_k + \eta_k$ where the matrix dimensions are similar. Hence, the model for the additional disturbances is given by

$$\eta_{k+1} = A_i \eta_k + \xi_k \qquad A_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.15}$$

where $\xi_k \sim N(0, Q_\xi)$ is a stochastic variable equivalent to $w_k$ and $v_k$ and taking $Q_{\xi_k} = I * 1^2$.

The offset-free model can be written by rephrasing the linearized continuous stochastic description of the system from equation (2.36) by tallying the new disturbance variable, giving a discrete model in the form of

$$X_{k+1} = AX_k + B(U_k + \eta_k) + E(D_k + w_k) \tag{3.16a}$$
$$\eta_{k+1} = A_i X_k + \xi_k \tag{3.16b}$$
$$Y_k = CX_k + v_k \tag{3.16c}$$
$$Z_k = C_z X_k \tag{3.16d}$$

Then, equation (3.16) can be rewritten in terms of the extended states and noises,

$$\breve{X}_k = \begin{bmatrix} X_k \\ \eta_k \end{bmatrix} \qquad \breve{w}_k = \begin{bmatrix} w_k \\ \xi_k \end{bmatrix} \tag{3.17}$$

therefore, the new stochastic model for offset-free control is

$$\breve{X}_{k+1} = A_e \breve{X}_k + B_e U_k + E_e D_k + G_e \breve{w}_k \tag{3.18a}$$
$$Y_k = C_e \breve{X}_k + v_k \tag{3.18b}$$
$$Z_k = C_{z,e} \breve{X}_k \tag{3.18c}$$

with the new matrices is given by

$$A_e = \begin{bmatrix} A & B \\ 0 & A_d \end{bmatrix} \quad B_e = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad E_e = \begin{bmatrix} E \\ 0 \end{bmatrix}$$
$$G_e = \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} \quad C_e = \begin{bmatrix} C & 0 \end{bmatrix} \quad C_{z,e} = \begin{bmatrix} C_z & 0 \end{bmatrix}$$

## 3.6 Summary

The concept of state estimation and the importance of estimating the current state in contributing the output predictions are briefly described in this chapter. Besides that, a brief description of the Kalman filter is included to have an overview of the filter as it is incorporated for state estimation and output predictions.

- Based on the model and the measurements, the current state of the MQT system was estimated and the algorithm from the static Kalman filter is able to compute the output predictions.
- The static Kalman filter is incorporated in MPC design and implementation for the MQT system since the covariance matrix, $P_{k|k-1}$ is kept constant and the computation of the controller is lightened due to the fact that covariance $P_{k+1|k}$ converges to equilibrium faster. This can be seen in Paper B where the comparison between the dynamic and static Kalman filter was presented.
- The stationary one-step ahead state error covariance matrix obtained from the Discrete-time Algebraic Riccati Equation (DARE) to complement the MPC design and implementation.
- An input disturbance model is introduced to model the impact of the immeasurable disturbance to achieve an offset-free control.

# CHAPTER 4

# Model Predictive Control

This chapter provides a demonstration of model predictive control implementation for the modified quadruple tank system. We give an overview of the MPC theoretically and describe the formulations mainly designed concurring with the MQT system. The demonstration of the controller implementation complete with the derivations of the equations including unconstrained MPC and input constrained MPC is presented in this chapter as exhibited in Paper C. Additionally, the input with soft output constrained MPC is included to further investigate the performance of the MPC. All results from the simulation works are compiled in the next chapter.

## 4.1 Unconstrained Model Predictive Controller

In this section, we develop unconstrained model predictive controllers based on the discrete-time state space models as in equation (3.1). Having designed the Kalman filter for the MQT system, the regulator will be implemented to form the complete MPC framework in Fig.1.3.

The control task is to track the setpoint trajectory as a quadratic optimization problem by developing an objective function that will minimize the deviation of the predicted output trajectory from the setpoint trajectory,

$$\min_{u} \quad \frac{1}{2}\sum_{k=1}^{N}||z_k - r_k||^2_{Q_z} + \frac{1}{2}\sum_{k=0}^{N-1}||\Delta u_k||^2_S \tag{4.1a}$$

s.t.

$$x_{k+1} = Ax_k + Bu_k + E_{d_k} \quad k = 0, 1, \dots, N-1 \tag{4.1b}$$

$$z_k = C_x x_k \quad k = 0, 1, \dots, N-1 \tag{4.1c}$$

From equation (2.46), $Z$ can be expressed in a matrix form of

$$Z_k = \Phi x_0 + \Gamma_u U + \Gamma_d D \tag{4.2}$$

where $Z_k$ and $R_k$ are

$$Z_k = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} \quad R_k = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix}$$

$U_k$, $R_k$ and $Z_k$ are deviation variables vectors. The weight matrices $Q_z$ and $S$ in equation (4.1) are realised as diagonal matrices since we want to penalise the deviation of

Tank 1 and Tank 2 from the desired targets, as well as large steps in the input variables, respectively.

In minimizing the objective function, equation (4.1) it can be expressed in a compact form as

$$min \quad \phi = \phi_z + \phi_{\Delta u} \tag{4.3}$$

The first term in the objective function is related to the desired target and the main part of the least squares minimization problem. It ensures that the system reaches towards the desired target values, $r_1$ and $r_2$.

$$\begin{aligned} \phi_z &= \frac{1}{2}||Z_k - R_k||^2_{Q_z} \\ &= \frac{1}{2}\left(\Phi x_0 + \Gamma U_k + \Gamma_d D - R_k\right)^2 \\ &\quad Q_z\left(\Phi x_0 + \Gamma U_k + \Gamma_d D - R_k\right) \end{aligned} \tag{4.4}$$

Let

$$b_k = R_k - \Phi x_0 - \Gamma_d D \tag{4.5}$$

and now we can express the problem as QP in minimizing $\phi_z$, an objective function based on the controlled variables.

$$\begin{aligned} \phi_z &= \tfrac{1}{2}\left(\Gamma U_k - b_k\right)^T Q_z \left(\Gamma U_k - b_k\right) \\ &= \tfrac{1}{2} U_k^T \Gamma^T Q_z \Gamma U_k - (\Gamma^T Q_z b_k)^T U_k + \rho \\ &= \tfrac{1}{2} U_k^T H_z U_k + g_z^T U_k + \rho \end{aligned} \tag{4.6}$$

where

$$H_z = \Gamma^T Q_z \Gamma \tag{4.7a}$$

$$\begin{aligned} g_z &= -\Gamma^T Q_z b_k \\ &= -\Gamma^T Q_z R_k + \Gamma^T Q_z \Phi_{x_0} + \Gamma^T Q_z \Gamma_d D \\ &= M_R R_k + M_{x_0} x_0 + M_d D \end{aligned} \tag{4.7b}$$

$$\rho = \frac{1}{2}\left(b^T Q_z b_k\right) \tag{4.7c}$$

Since using the objective function based only on the controlled variables is insufficient, we include the input variables, $\phi_{\Delta u}$ in the objective function as the second term. The second term is the regularization term, which ensures smooth input solutions which minimize the difference of $u_k$ from the previous input to have less error.

$$min \quad \phi_{\Delta u} = \frac{1}{2}\sum_{k=0}^{N-1} ||\Delta u_k||^2_S \tag{4.8}$$

We want to rewrite this problem into standard QP. First, we want to derive $\phi_{\Delta u}$ using similar approach to $\phi_z$ but $\Delta u$ needs to be expressed in terms of $u_k$,

$$\Delta u_k = \begin{bmatrix} \Delta u_0 \\ \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 - u_0 \\ u_2 - u_1 \\ u_3 - u_2 \end{bmatrix} - \begin{bmatrix} u_{-1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.9}$$

To have better formulation, we introduce $\Lambda$, $U_k$ and $I_0$ as

$$\Lambda = \begin{bmatrix} I & 0 & 0 & 0 \\ -I & I & 0 & 0 \\ 0 & -I & I & 0 \\ 0 & 0 & -I & I \end{bmatrix} \quad U_k = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad I_0 = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $I$ denotes an identity matrix of the size of $u$, $I_0$ denotes the block vector with $I$ in the first entry while having zero matrices fill up the rest of the rows to have the same row-size as matrix $\Lambda$, then $\Delta u_k$ can be rewritten as

$$\Delta u_k = \Lambda U_k - I_0 u_{-1} \tag{4.10}$$

We expand equation (4.8) by substituting with equation (4.10) and the objective function $\phi_{\Delta u}$ yields

$$\begin{aligned} \phi_{\Delta u} &= \tfrac{1}{2}||\Lambda U_k - I_0 u_{-1}||_S^2 \\ &= \tfrac{1}{2}U_k^T \left(\Lambda^T \bar{S} \Lambda\right) U_k + \left(-\Lambda^T \bar{S} I_0 u_{-1}\right)^T U_k + \rho \\ &= \tfrac{1}{2}U_k^T H_{\Delta_u} U_k + g_{\Delta_u}^T U_k + \rho \end{aligned} \tag{4.11}$$

where

$$H_{\Delta_u} = \Lambda^T \bar{S} \Lambda \tag{4.12a}$$

$$\begin{aligned} g_{\Delta_u} &= -\Lambda^T \bar{S} I_0 u_{-1} \\ &= M_{u-1} u_{-1} \end{aligned} \tag{4.12b}$$

$$\rho = \frac{1}{2}\left(U_k^T I_0^T \bar{S}\right) \tag{4.12c}$$

Combining equation (4.6) and (4.11) and $\rho$ is disregarded as these are constant, resulting an optimization of the deviation from the setpoint (reference trajectory) and the input variables given as equation (4.1) becomes

$$\min_{u} \quad \frac{1}{2}U_k^T H U_k + g^T U_k \tag{4.13a}$$

s.t.

$$x_{k+1} = Ax_k + Bu_k + E_{d_k}, \quad k = 0, 1, \ldots, N-1 \tag{4.13b}$$

$$z_k = C_x x_k, \quad k = 0, 1, \ldots, N \tag{4.13c}$$

where

$$H = H_z + H_{\Delta_u} \qquad g = g_z + g_{\Delta_u} \tag{4.14}$$

MPC computation when stated as a QP is solved by first computing

$$g = M_R R_k + M_{x_0} x_0 + M_d D + M_{u-1} u_{-1} \tag{4.15}$$

then solving the QP

$$\begin{aligned} u^* &= -H^{-1} g \\ &= -H^{-1}\left(M_R R_k + M_{x_0} x_0 + M_d D + M_{u-1} u_{-1}\right) \\ &= L_R R + L_{x_0} x_0 + L_d D + L_{u-1} u{-1} \end{aligned} \tag{4.16}$$

with the first block row of $L_R, L_{x_0}, L_d, L_{u-1}$ is given by $K_R, K_{x_0}, K_d, K_{u-1}$ and the optimal control law is

$$u_0^* = K_R R + K_{x_0} x_0 + K_d D + K_{u-1} u{-1} \tag{4.17}$$

## 4.2 Constrained Model Predictive Controller

An optimal control decision for unconstrained MPC could affect the flow of the liquid. It could be applying negative flows into the two controllable pumps, giving an impact of sucking the liquid out of the tanks. It also could be inviable changes in the flow rates due to potential mechanical limitations or the rate at which the control input can be changed is bounded. Therefore, to avoid infeasible decisions by the controller and to ensure that the MQT system works under a secure condition without damaging any mechanical parts, we introduced the constraints to allows the controller to decide an optimal control moves within certain bounds. MPC is well recognized for this advantages and we will show the implementation of MPC with constraints classified as input and soft output constraints.

### 4.2.1 Input Constrained MPC

To make the simulation of the model-based controller on the MQT system more realistic, we considered two different hard constraints; first is the one that sets the upper and lower bounds on the manipulated variables, $u_{min} \leqslant u \leqslant u_{max}$ and the other one is the rate of change in input, $\Delta u_{min} \leqslant \Delta u \leqslant \Delta u_{max}$. The formulation of the problem is the same as stated in equation (4.1) but subjected to constraint it becomes

$$\min_{u} \quad \frac{1}{2}\sum_{k=1}^{N} ||z_k - r_k||^2_{Q_z} + \frac{1}{2}\sum_{k=0}^{N-1} ||\Delta u_k||^2_S \tag{4.18a}$$

s.t.

$$x_{k+1} = Ax_k + Bu_k + E_{d_k} \quad k = 0, 1, \ldots, N-1 \tag{4.18b}$$

$$z_k = C_x x_k \quad k = 0, 1, \ldots, N-1 \tag{4.18c}$$

$$u_{min} \leqslant u_k \leqslant u_{max} \quad k = 0, 1, \ldots, N-1 \tag{4.18d}$$

$$\Delta u_{min} \leqslant \Delta u_k \leqslant \Delta u_{max} \quad k = 0, 1, \ldots, N-1 \tag{4.18e}$$

In standard QP form, this can be written as (4.13) and the inequality constraints referring to equation (4.18e) needs to be updated at each iteration since it contains the input variables form the previous step, $u_{k-1|k}$. In developing the simulation code for the input constraint MPC, the Hessian matrix is obtained by offline computations, essentially the same as the unconstrained MPC but the calculation of the inequality matrix and the corresponding upper and lower bounds are set beforehand. Likewise, during the regulator process, equation (4.18d) is supplied as upper and lower bounds to the quadratic solver in Matlab and the 2 inequalities in equation (4.18e) are formulated in the form of

$$\begin{bmatrix} \Delta u_{min} \\ \Delta u_{min} \\ \vdots \\ \Delta u_{min} \end{bmatrix} \leqslant \begin{bmatrix} u_0 - u_{-1} \\ u_1 - u_0 \\ \vdots \\ u_N - u_{N-1} \end{bmatrix} \leqslant \begin{bmatrix} \Delta u_{max} \\ \Delta u_{max} \\ \vdots \\ \Delta u_{max} \end{bmatrix} \tag{4.19}$$

Since the first row contains $u_{-1}$ this can be written as

$$\Delta u_{min} + u_{-1} \leqslant u_0 \leqslant \Delta u_{max} + u_{-1} \tag{4.20a}$$

$$u_{min} \leqslant u_0 \leqslant u_{max} \tag{4.20b}$$

and the rest of the rows are arranged in the form of

$$\begin{bmatrix} \Delta u_{min} \\ \Delta u_{min} \\ \vdots \\ \Delta u_{min} \end{bmatrix} \leqslant \begin{bmatrix} -I & I & & & \\ & -I & I & & \\ & & \ddots & \ddots & \\ & & & -I & I \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \leqslant \begin{bmatrix} \Delta u_{max} \\ \Delta u_{max} \\ \vdots \\ \Delta u_{max} \end{bmatrix} \tag{4.21}$$

which can be simplified as

$$\Delta U_{min} \leqslant \Lambda U_k \leqslant \Delta U_{max}$$

Therefore, an optimization of the deviation from the setpoint (reference trajectory) and the input variables becomes

$$\min_{u} \quad \Phi = \frac{1}{2} U_k^T H U_k + g^T U_k \tag{4.22a}$$

s.t.

$$U_{min} \leqslant U_k \leqslant U_{max} \tag{4.22b}$$

$$\Delta U_{min} \leqslant \Lambda U_k \leqslant \Delta U_{max} \tag{4.22c}$$

where $H$ and $g$ is as given in (4.14). The optimised input is returned and the first two entries obtained and applied in the next iteration, similar to the previous unconstrained MPC.

### 4.2.2 Input and Soft Output Constrained MPC

Since both unconstrained and input constrained MPC is derived in the previous sections, we also need to include the output constraint. A feasible approach to introduced any constraints on the controlled variables is to directly add hard constraints as shown in section 4.2.1

$$z_{min,k} \leqslant z_k \leqslant z_{max,k} \tag{4.23}$$

Generally, the output constraints should not be violated, but due to unknown disturbances and noises, it could be violated and the QP would have no feasible solution which can cause an abrupt stop to the controller. The more sophisticated way of introducing constraints on the controlled variables is to ensure that the QP is able to yield a solution to the controller by adding a slack variable, $\eta_k$ to formulate the soft constraints,

$$\begin{aligned} & z_k \leqslant z_{max,k} + \eta_k && k = 1, 2, ...N \\ & z_k \leqslant z_{min,k} - \eta_k && k = 1, 2, ...N \\ & \eta_k \geqslant 0 && k = 1, 2, ...N \end{aligned}$$

significantly by introducing a penalty term in the objective function to allow the output constraints to be violated should the physical constraints are exceeded,

$$min \quad \phi_s = \frac{1}{2} \sum_{k=1}^{N} ||\eta_k||^2_{S_\eta} + s_\eta^T \eta_k \tag{4.24}$$

Therefore, the input constrained and soft output constrained MPC can be formulated as

$$\min_{u} \quad \frac{1}{2}\sum_{k=1}^{N}||z_k - r_k||^2_{Q_z} + \frac{1}{2}\sum_{k=1}^{N}||\eta_k||^2_{S_\eta} + s_\eta^T\eta_k + \frac{1}{2}\sum_{k=0}^{N-1}||\Delta u_k||^2_S \tag{4.25a}$$

s.t.

$$x_{k+1} = Ax_k + Bu_k + E_{d_k} \quad k = 0,1,\ldots,N-1 \tag{4.25b}$$

$$z_k = C_x x_k \quad k = 0,1,\ldots,N \tag{4.25c}$$

$$u_{min} \leqslant u_k \leqslant u_{max} \quad k = 0,1,\ldots,N-1 \tag{4.25d}$$

$$\Delta u_{min} \leqslant \Delta u_k \leqslant \Delta u_{max} \quad k = 0,1,\ldots,N-1 \tag{4.25e}$$

$$z_k \leqslant z_{max,k} + \eta_k \quad k = 1,2,...N \tag{4.25f}$$

$$z_k \leqslant z_{min,k} - \eta_k \quad k = 1,2,...N \tag{4.25g}$$

$$\eta_k \geqslant 0 \quad k = 1,2,...N \tag{4.25h}$$

We want to rewrite the objective function into QP form and the new term in the objective function from equation (4.24) is formulated as

$$\phi_s = \frac{1}{2}\eta^T H_\eta \eta + g_\eta \eta \tag{4.26}$$

by introducing

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{bmatrix} \tag{4.27}$$

where the Hessian and gradient are

$$H_\eta = \begin{bmatrix} H_\eta & & \\ & \ddots & \\ & & H_\eta \end{bmatrix} \qquad and \qquad g_\eta = \begin{bmatrix} g_\eta \\ \vdots \\ g_\eta \end{bmatrix} \tag{4.28}$$

Equation (4.25) can be augmented and rewritten into matrix notation such as

$$min \quad \phi = \frac{1}{2}\begin{bmatrix} U \\ \eta \end{bmatrix}^T \begin{bmatrix} H & 0 \\ 0 & H_\eta \end{bmatrix} \begin{bmatrix} U \\ \eta \end{bmatrix} + \begin{bmatrix} g \\ g_\eta \end{bmatrix}^T \begin{bmatrix} U \\ \eta \end{bmatrix} \tag{4.29}$$

subject to lower and upper bounds

$$\begin{bmatrix} U_{min} \\ 0 \end{bmatrix} \leqslant \begin{bmatrix} U \\ \eta \end{bmatrix} \leqslant \begin{bmatrix} U_{max} \\ \infty \end{bmatrix} \tag{4.30}$$

and subject to constraints for input rate and output in the following formulation

$$\begin{bmatrix} \Delta U_{min} \\ -\infty \\ \bar{Z}_{min} \end{bmatrix} \leqslant \begin{bmatrix} \Lambda & 0 \\ \Gamma & -I \\ \Gamma & I \end{bmatrix} \begin{bmatrix} U \\ \eta \end{bmatrix} \leqslant \begin{bmatrix} \Delta U_{max} \\ \bar{Z}_{max} \\ \infty \end{bmatrix} \tag{4.31}$$

Then to have better formulation we simplified the system by introducing $\bar{U}$, $\bar{H}$ and $\bar{g}$ in the form of

$$\bar{H} = \begin{bmatrix} H & 0 \\ 0 & H_\eta \end{bmatrix} \quad \bar{U} = \begin{bmatrix} U \\ \eta \end{bmatrix} \quad \bar{g} = \begin{bmatrix} g \\ g_\eta \end{bmatrix}$$

The lower and upper bound can be simplified as $l \leqslant \bar{U} \leqslant u$ and the contraints for input rate and output can be simplified as $b_l \leqslant A\bar{U} \leqslant b_u$. The standard QP is given as

$$\min_{u} \quad \Phi = \frac{1}{2}\bar{U}^T\bar{H}\bar{U} + \bar{g}^T\bar{U} \tag{4.32a}$$

s.t.

$$l \leqslant \bar{U} \leqslant u \tag{4.32b}$$

$$b_l \leqslant A\bar{U} \leqslant b_u \tag{4.32c}$$

## 4.3 Summary

One of the main objectives of the study is presented in this chapter. The implementation of Model Predictive Control for the Modified Quadruple Tank system is developed and demonstrated through simulations based on the state space model derived in the previous chapters.

- The first part of the chapter deals with the formulation of the unconstrained MPC based on the linear discrete-time state space model. Then the MPC regulation problem which is formulated as a quadratic optimization problem is illustrated. The associated quadratic optimization problem is solved explicitly and all this explanation is disclosed in Paper C.
- The second part of the chapter includes the constraints. The formulation of different constrained model predictive controllers is considered and the quadratic optimization problems resulting from the constrained MPC regulation problems.
- The results and analysis of these MPC algorithms are presented in the next section.

# CHAPTER 5

# Simulations and Analysis

This chapter shows the closed-loop simulation results and analysis from the MPC implementation for the modified quadruple tank system. MPCs presented in the following sections are discussed and evaluated with various test cases. The main objective is to evaluate the performance of the MPC in terms of the behaviour of the system and to verify should the realisations are physically feasible. The chapter is organized as follows. The first part of the chapter presents the simulation results from the Unconstrained MPC and the second part is followed by the constrained MPCs. As for the constrained MPCs, the section will be divided into two parts, Input Constrained MPC and Input and Soft Output Constrained MPC. All results from the simulation are compiled and presented in this chapter.

## 5.1 Overview

MPC algorithm constructed in Chapter 4 are implemented and tested with several experiments and the performances are analysed based on the capability of the algorithm of the controller to compensates the disturbances by modifying the inputs such that the outputs remain close to the reference. In other words, the performance of the controller is measured based on the observation of the output responses for setpoint tracking. Each experiment is analyzed with different level of percentage in step input change, however by using 15% step input change resulting in a perceptible responses suitable for this study. For simulation purposes, the linear stochastic model is utilized in which noise is included as a normal distribution to the disturbance variables and the measurement variables.

Fig.5.1 shows the flowchart of the simulation work for MPC implementation. Starts with defining the parameters of the MQT system in *Parameter.m* and in the *MPCControl.m* command window, a Matlab function named *MPCDesign.m*, *MPCInput.m* and *MPCSimLin.m* is developed.

Within the *MPCDesign.m* function, the Hessian matrices for the corresponding optimisation problem are built in *DesignMPC_Matrices* by taking the discrete-time linear model matrices, condensed LTI matrices and weights of the objective function as inputs specified in *MPCInput.m*. The discrete-time linear model matrices is obtained from the continuous-time non-linear model that has been linearized and discretized beforehand.

In addition, the filter gains of the static Kalman filter that are precomputed separately in the function *DesignKalman* is pass to *MPCSimLin.m* through *MPCDesign.m*. At every stage of the closed-loop, the main task of *MPCCompute* function is to look for a solution of QP sub-problem solved with the aid of Matlab utility, *QPSolver*. Finally, the variables were transformed from deviation to physical variables and graphs are illustrated in *MPCPlot.m*.

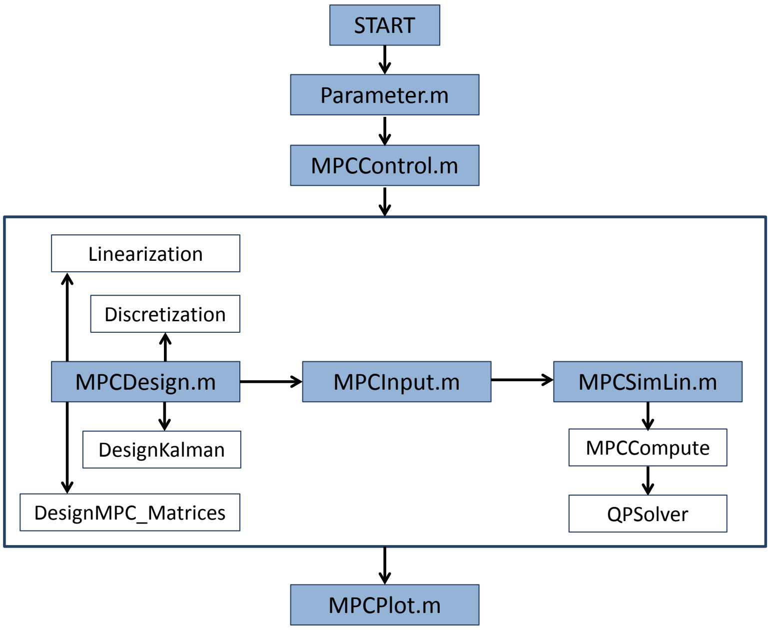



**Figure 5.1:** Flowchart of the simulation work for MPC implementation

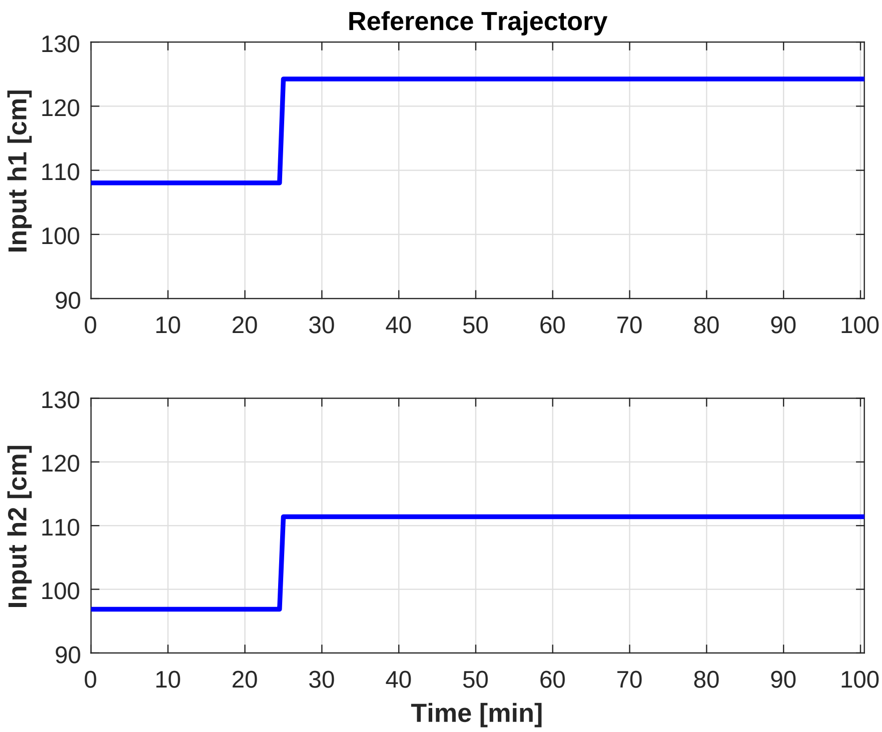



**Figure 5.2:** 15% step input change in reference trajectory of $F_1$ and $F_2$ for **Experiment 1** and **Experiment 2**

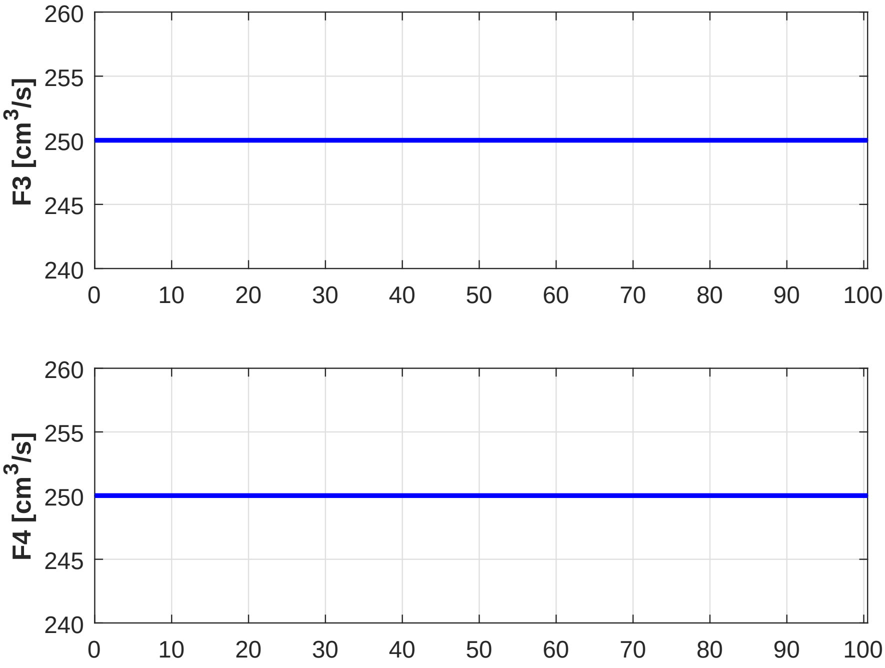



**Figure 5.3:** No step input change in $F_3$ and $F_4$ for **Experiment 1**

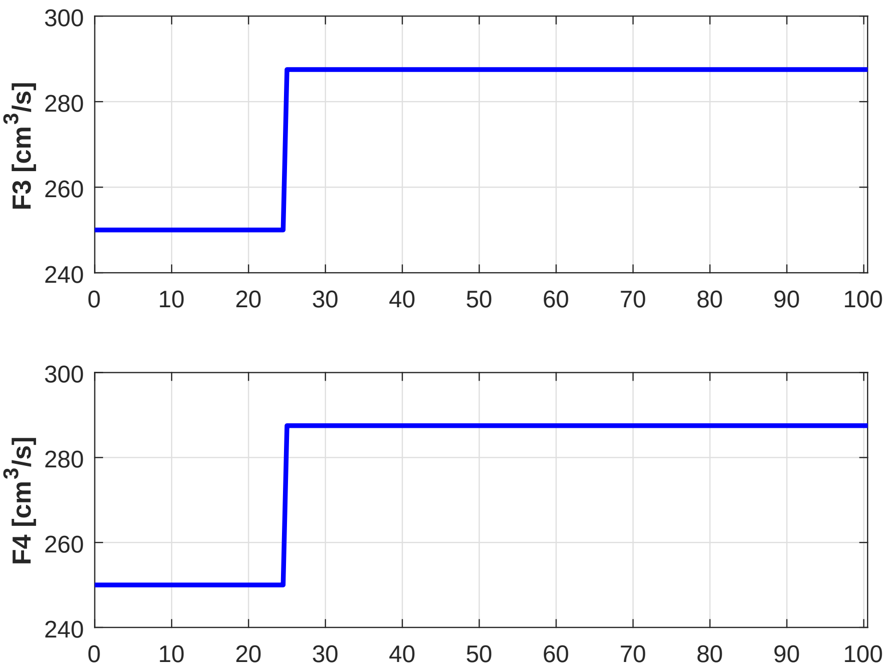



**Figure 5.4:** 15% step input change in $F_3$ and $F_4$ for **Experiment 2**

A total of two experiments are carried out for each MPC approaches, for **Experiment 1** step input change in $F_1$ and $F_2$ is considered and as for **Experiment 2**, additional step input change in the disturbances of $F_3$ and $F_4$ is included. Results and analysis of the simulations for each MPCs which was run for 100 minutes of simulation time is discussed below. The step input changes is initiate after 25 minutes as shown in Fig.5.2 for the reference trajectory and as for the disturbances for **Experiment 1** and **Experiment 2** is as shown in Fig.5.3 and Fig.5.4 respectively.

As for the Unconstrained MPC, the associated quadratic optimization problem is solved explicitly. On the other hand, the second part of the chapter deals with the problem associated with bounds and constraints in the tanks system subject to the limitations in the instrumentation, such as the specifications of valves, pumps performance and/or tanks capacity. These bounds and constraints need to be assimilated and adapted to the algorithm to have a realistic application.

In this part of the chapter, the formulation of different constrained model predictive controllers and the quadratic optimization problems resulting from the constrained MPC regulation problems is considered.

## 5.2 Closed-loop Simulation for Unconstrained MPC

In this part of the study, the Unconstrained MPC is implemented to the MQT system in a way to manifest the function and behaviour of an MPC. It is expected that the implementation of MPC will drive the system to the set point and the solution to be as smooth as possible. Moreover, the performance of a first-hand straightforward MPC can be evaluated. These objectives can be achieved empirically by testing the algorithm presented in section 4.1 on the above described conditions, influenced by white process noise and measurement noise.

Fig.5.5 and Fig.5.6 depicts the results of the closed-loop simulation for the Unconstrained MPC for **Experiment 1** and **Experiment 2** respectively. The top two graphs are the output responses and the bottom two presents the obtained inputs $u_1$ and $u_2$ from the MPC. The red line on the output responses indicates the target values for the output variables and the black line illustrates the measured output.

In **Experiment 2**, the disturbance variables are initialised such that they contain a 15% step input change in order to evaluate the performance of the regulator in compensating for the upcoming disturbance.

Primarily, from these figures, it shows that the linearized model is a good representation of the system and MPC is well demonstrated. Both output responses in **Experiment 1** and **Experiment 2** shows that the MPCs are able to handle the changes when a new setpoint is introduced and able to reach the new value relatively fast.

The responses show that the system is able to track the references with minimal overshoot and small transient deviations, keeping the desired height levels $h_1$ and $h_2$ of Tank 1 and Tank 2 respectively at the desired set points. The algorithm compensates the step input change and the step change in disturbances by reducing the inputs to keep the water levels at the desired reference values.

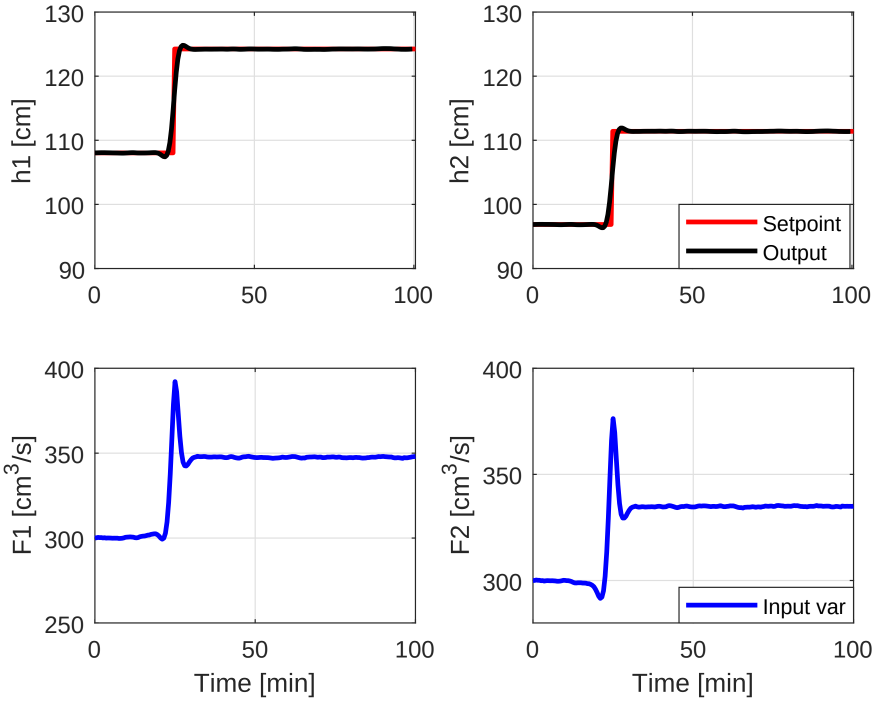



**Figure 5.5:** Unconstrained MPC for **Experiment 1**

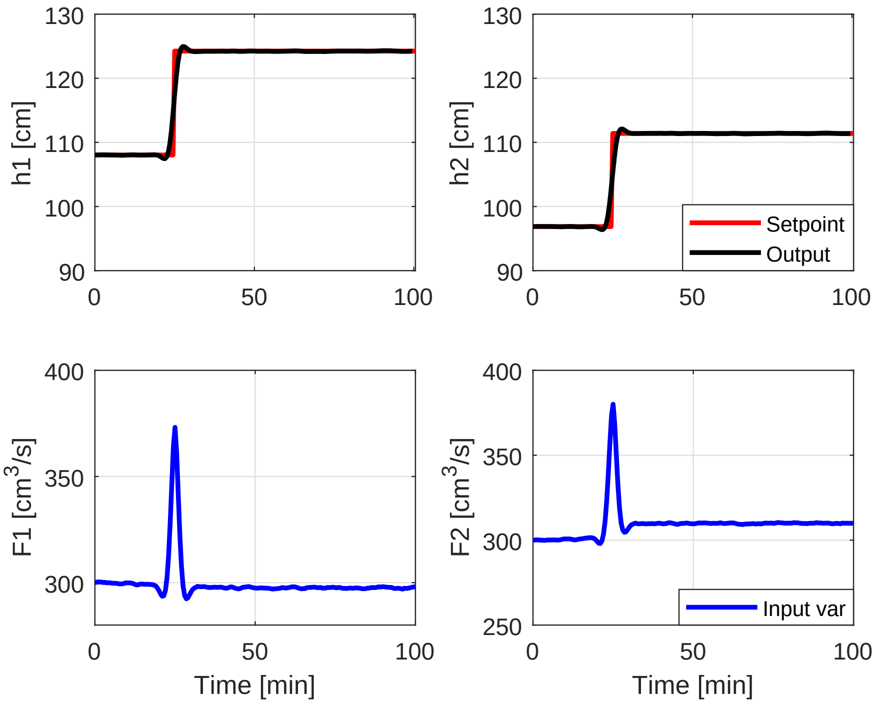



**Figure 5.6:** Unconstrained MPC for **Experiment 2**

Although the MPC managed to compensate the sudden change in inputs and disturbances, note that in order to achieve this circumstance, an abrupt and sharp increase in the input variables $F_1$ and $F_2$ occurred in both experiments which possibly would be infeasible for real applications. This glitch can be solved by implementing the subsequent MPC strategy in the next section.

## 5.3 Closed-loop Simulation for Constrained MPC

From the previous section, the Unconstrained MPC is able to keep the desired set point of levels of height in Tank 1 and Tank 2 and compensate for the disturbances. However, it can be observed that even though the outputs are driven to the reference in all cases, the inputs present a very high and steep top when there is a sudden change in the reference, indicates that the water is injected almost instantaneously. This occasion is unacceptable from the practical point of view and can be stifled by considering the limitations of the tank system.

Therefore, we would like to incorporate some restrictions in both input and output limits, appear as bounds and equality/inequality constraints in the objective of the optimization problem as described in section 4.2.

In this section, the Input Constrained MPC and a combination of Input and Soft Output Constrained is presented and the performance is observed from identical experiments in the previous section. Each MPC approaches is tested with a different combination of the boundary of the constraints. Elaboration of the details of the boundary of the constraints is presented in the designated sections.

### 5.3.1 Input Constrained MPC

A similar closed loop simulation was implemented and **Experiment 1** and **Experiment 2** is repeated but in this section, the constraints for the input is included. In practice, the pumps operates with maximum and minimum flow, $[cm^3/s]$ and rate of change in the flow, $[cm^3/s^2]$. The considered constraints for **Experiment 1** is given as

$$\begin{array}{rclll} 0 & \leqslant & u_{min} & \leqslant 350 & (cm^3/s) \\ -20 & \leqslant & \Delta u_k & \leqslant 20 & (cm^3/s^2) \end{array}$$

and for **Experiment 2** is given as

$$\begin{array}{rclll} 0 & \leqslant & u_{min} & \leqslant 310 & (cm^3/s) \\ -20 & \leqslant & \Delta u_k & \leqslant 20 & (cm^3/s^2) \end{array}$$

Fig.5.7 and Fig.5.8 represents the output of the simulation in response to a set point change with the same condition of reference trajectory and disturbances as described earlier in **Experiment 1** and **Experiment 2** respectively.

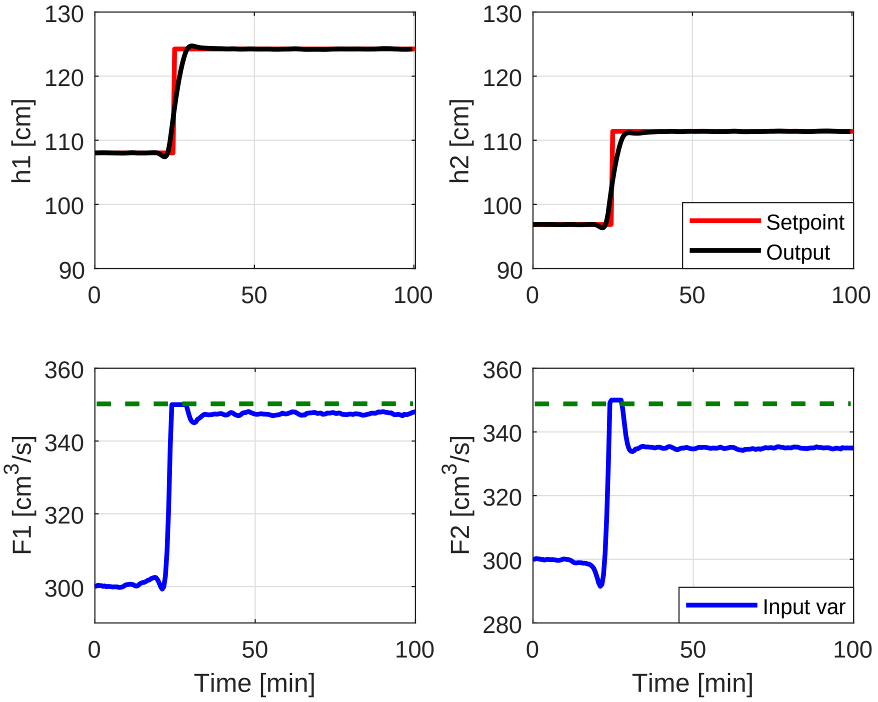



**Figure 5.7:** Input constrained MPC for **Experiment 1**

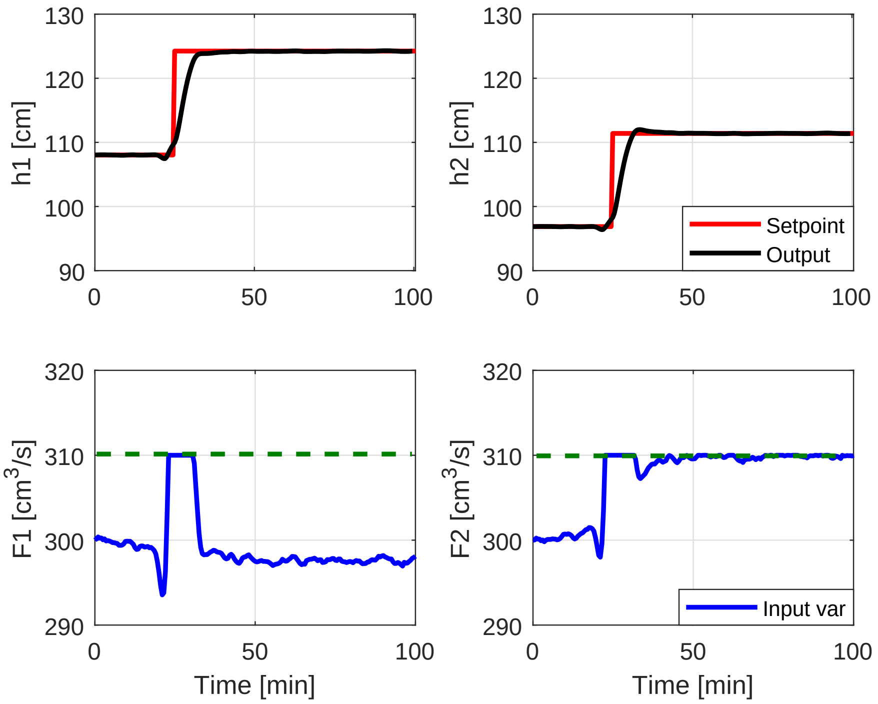



**Figure 5.8:** Input constrained MPC for **Experiment 2**

From both experiments, it can be seen that the output responses of $h_1$ and $h_2$ are able to track the new set point with satisfactory performance although the transient properties are slightly deteriorated due to disturbance compensations from the input constraints and the capacity of the tank system. Consequently, it is physically impossible to stream the water to the tanks instantaneously so that the output responses able to reach the reference trajectory momentarily.

Referring to the flow of $F_1$ and $F_2$ for both experiments, it is noticeable that the water flow is within the boundaries but due to the constraints, it affects the flow by preventing the development of the abrupt peaks and causing a plateau for smoother flow, indicating that the rate constraints results in a more well-behaved flow characteristics with a slight loss of transient properties and the algorithm compensates the peaks in the disturbances by decreasing the inputs. Note that the constraints for maximum flow for **Experiment 1** ($350cm^3/s$) is higher than for **Experiment 2** ($310cm^3/s$). These values are chosen based on the flow shown in Fig.5.5 and Fig.5.6 in order to eliminate the sharp increase accordingly.

With input constraints, the controller is capable of operating within the limit bounds but with an acceptable drawback of transient response in the outputs and the amount of computation required is higher compared to the unconstrained controller. Next, we would like to include the soft output constraint in the algorithm to have more stringent outputs to signify the real application.

### 5.3.2 Input and Soft Output Constrained MPC

In real experiments, each tank has a certain limit of capacity to hold the water while the water flows in and out of the tanks, depending on the process. The capacity of the tanks should be taken into consideration to avoid overflow. Problems might occur should the reference value is given to the algorithm such that the level of the water in Tank 1 and Tank 2 exceeded the maximum level. Therefore, this limitation is incorporated in the optimization sub-problem, act as a safeguard for the algorithm to produce feasible outputs if given any reference trajectory that drives the outputs above the threshold.

In this section, the closed-loop simulation was implemented similarly to the Input Constrained MPC except, in this case, we want to further augment the criterion function by introducing the soft constraints to the regulator as shown in section 4.2.2. **Experiment 1** and **Experiment 2** is duplicated from the previous section but with rate of change in the flow, $[cm^3/s^2]$ given as

$$-10 \quad \leqslant \quad \Delta u_k \quad \leqslant 10 \quad (cm^3/s^2)$$

and additional soft output constraints as in equation 4.23 given as

$$0 \quad \leqslant \quad z_k \quad \leqslant 12$$

In **Experiment 2**, the maximum and minimum flow, $[cm^3/s]$ is given as

$$0 \quad \leqslant \quad u_{min} \quad \leqslant 300 \quad (cm^3/s)$$

For **Experiment 1**, the results is presented in Fig.5.9 and for **Experiment 2**, the result is presented in Fig.5.10. It is noticeable that on the output responses, the green dotted lines represent the soft constraints on the target values, giving $r_{max} = 120$ and $r_{max} = 109$ for $h_1$ and $h_2$ respectively.

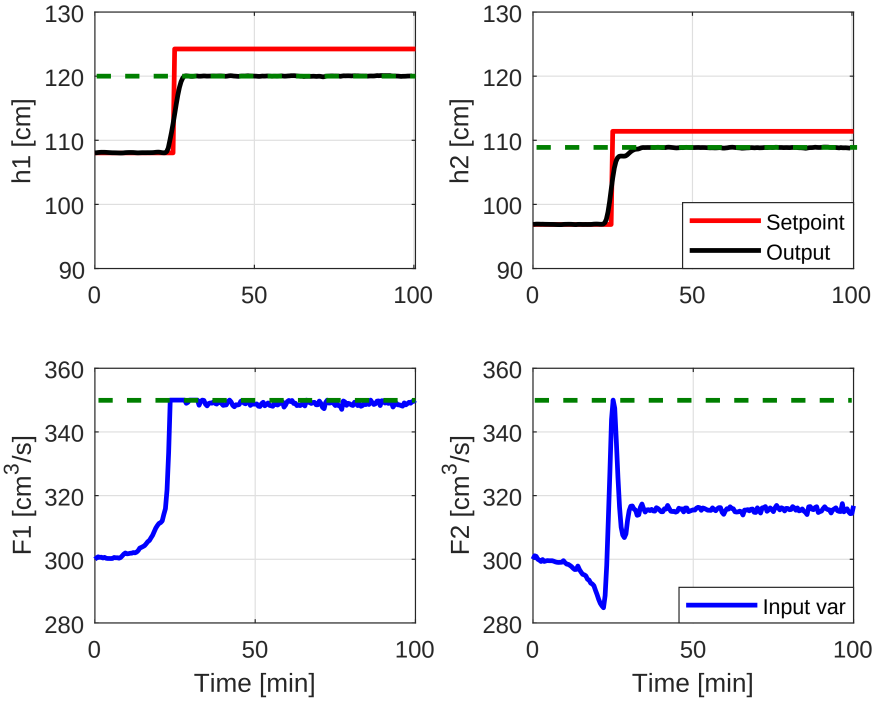



**Figure 5.9:** Input and soft output constrained MPC for **Experiment 1**

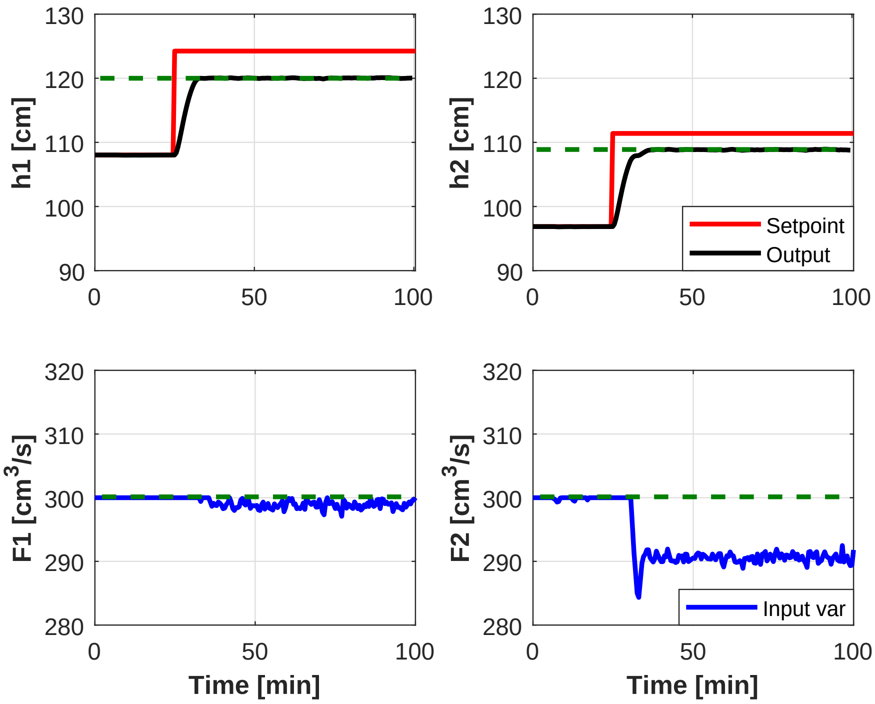



**Figure 5.10:** Input and soft output constrained MPC for **Experiment 2**

From observation, the output variables stay within the boundary of the soft constraint, remains below the threshold set by the capacity of the tank even though the reference is higher and able to track the setpoint changes in an acceptable performance of transient response. This indicates that the soft output constraint utilized act as a safeguard where it realizes a relaxation around the reference value while keeping the system from exceeding the limits.

Additionally, referring to the first experiment in Fig.5.9, the flow of $F_1$ is within the boundaries and rather smoother compared to the previous approach which is the Input Constrained MPC, but the control algorithm could not suppress the step input in the reference trajectory for the flow of $F_2$, resulting a sharp peak making it infeasible for real application. This could be different should the upper boundary of the flow, $u_{min}(cm^3/s)$ is given with a lower value particularly for input $F_2$ to compensate with the changes of input without changing the upper boundary of the flow in $F_1$. This behaviour can also be seen in the second experiment in Fig.5.10 but since the upper boundary of the flow is lower than the first experiment ($300cm^3/s$), the controller managed to eliminate the sharp peak and not violating the limit of the soft output constraint. Besides, the element of oscillation is more noticeable in the flow of input variables, $F_1$ and $F_2$ in comparison to the previous approaches but the output responses in $h_1$ and $h_2$ shows good performance of transient response and very smooth output responses.

## 5.4 Summary

This chapter is the continuity of the MPC algorithms described in the previous chapter. Three MPC algorithms namely Unconstrained MPC, Input Constrained MPC and Input and Soft Output Constrained MPC is tested with two experiments, each with a different condition of inputs and disturbances and the results is demonstrated and analysed in this chapter.

- The overview of the closed loop simulation and analysis of the MPC demonstration is presented in the first part of the chapter. The condition of the experiments executed and the illustration of the Matlab simulation is explained.
- In the second part of the chapter, the results for the Unconstrained MPC is presented and discussed.
- The final part of the chapter deals with the constrained MPC and it is divided into two subsections. The importance of these constraints is clarified and the boundaries of the constraints are specified. First, the Input Constrained MPC is presented complete with results and analysis followed by the Input and Soft Output Constrained MPC.

CHAPTER 6

# Conclusion

In this chapter, the conclusion of the thesis is outlined based on the objective of the study. We featured the conclusions from each phase of the research project which we highlight and summarizes the important work done and significant results.

In this thesis, the Modified Quadruple Tank (MQT) system is studied and modelled then the Model Predictive Control (MPC) strategy is implemented and demonstrated. In that effort, we developed the deterministic and stochastic linear discrete-time state space models for the tank system and derived the static Kalman filter algorithm for state estimation. The filtered part of the state estimation is used to estimates the current state whilst the prediction part is used to predict the future output trajectory. The models of the MQT system and the state estimation from the Kalman filter were used to facilitate the development and comparison of MPC approaches. The importance of the constraints that represent the physical limitations of the system is described and the comparison between unconstrained and constrained MPC is presented and discussed. These controller strategies and the bounds of the constraints were tailored for the MQT system for the output responses to smoothly reach the set point given a new reference trajectory.

## 6.1 Modeling of Modified Quadruple Tank System

In this first phase, the modified quadruple tank system is presented, a virtual plant of the system is created and the dynamics of the system is modelled as in Chapter 2. A simple first-principle model was developed which described by a non-stiff Ordinary Differential Equations (ODE). The model comprises of deterministic and stochastic non-linear model is simulated and then the steady state analysis is investigated in order to identify the transfer function from the normalized step responses and to develop an operating window for possible selection of set points.

The transfer functions are estimated from the normalized step responses with 10% step increment of input. This is due to the fact that the responses are the closest to the actual output responses. An accuracy test was done for noise estimation and by comparison, the values of variances identified from the transfer functions yield accurate estimation and the mean values are approximated as fairly accurate. These values are tabulated for comparison purposes. Subsequently, an analysis of Markov parameters is executed to determine the sampling time. The discrete-time linear state-space model was calculated using an MPC toolbox and the impulse response of the system was obtained. The results show that the step response from the identified transfer functions and the approximated state space models are identical, validating the estimation.

Next, the Linear Discrete-time State Space model representation was realized to make it suitable for computer control analysis. The model was linearized and discretized and then the structure of the model was reconstructed by introducing the Markov parameters to have an observer canonical form with minimal realization. The Jacobian matrix formation is used by applying the first-order term only of Taylor expansion for linearization and assuming the zero-order-hold (ZOH) of the variables at specified sampling points is applied for discretization. As for the conclusion, a model that represents the dynamic of the modified quadruple tank system in the form of Linear Discrete-time is methodically realized and set for the next phase.

## 6.2 State Estimation for the Discrete-Time Linear System

A Kalman filter algorithm for state estimation is developed in this part of the thesis. The rationale and importance of the estimation are highlighted in Chapter 3 where the state estimation is provided to the estimator part of the MPC and to accomplish this, a static Kalman filter is incorporated. The filter's algorithm is derived from the discrete-time linear model obtained in the previous chapter. The feature of the Kalman filter is that it minimizes the sum of squared errors between the current state and the state estimation. Thus, the current state of the MQT system was estimated and the output prediction was computed from the static Kalman filter's algorithm based on the model and the measurements.

The coefficients are computed off-line and the stationary one-step ahead state error covariance matrix, $P$ was obtained from the Discrete-time Algebraic Ricatti Equation (DARE). Then a simulation was done and the measurement response was compared to observe the role of the static Kalman filter. In conclusion, the prediction is estimated and the simulated measurements and outputs are predicted. Therefore, the Kalman filter is considered for MPC implementation in the next chapters.

## 6.3 Development and Simulation of Model Predictive Control

The essence of the study lies in this final phase of the research project. The MPC algorithms were developed and implemented for the MQT system. The aim is for the controller to be able to track the setpoint trajectory as a quadratic optimization problem. The objective function was developed in order to minimize the deviation of the predicted output trajectory from the setpoint trajectory. The predictions part of the static Kalman filter was utilized by the constraint regulator and the future output trajectory was predicted. The formulations for MPC is derived and presented in detail in Chapter 4 and the results from the simulation from a number of experiments are discussed in Chapter 5. The unconstrained MPC was demonstrated to apprehend the operation MPC without any restrictions and then followed by the constrained MPC to disclosure its versatility.

In unconstrained MPC, the changes in reference trajectory are successfully tracked relatively fast with small transient deviations but impractical for real application due to an instantaneous sharp increase in the flow of input variables. Hence, constraints were included in the objective function of the optimization problem to overcome this issue. First, an input constraint was introduced and then a combination of input and soft output constraints were included. With input constraint, even though the transient response is insignificantly degraded, the MPC was well performed, the new set point was efficiently tracked and the sharp peak in the flow of input variables was successfully suppressed. It would be more effective if the boundaries of flow of input variables $U_{min}$ can be individually selected as $U_{min_1}$ and $U_{min_2}$ designated for $F_1$ and $F_2$ respectively for the system to have smoother response.

Finally, to avoid overflow in the tanks, the capacity of the tanks is taken into consideration and formed as a soft output constraint. As expected, the output responses operated normally and smoothly within the output boundaries and the input constraints were not violated, however, the flow of the input variables discloses oscillation due to the stringent condition of operation.

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# Part II

# Published Journal and Conference Papers

PAPER A

# Modeling and simulation of a modified quadruple tank system

**Authors:**

Sazuan Nazrah Mohd. Azam and John Bagterp Jørgensen

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# Modeling and Simulation of a Modified Quadruple Tank System

Sazuan Nazrah Mohd. Azam
Department of Applied Mathematics and Computer Science
Technical University of Denmark
DK-2800 Kgs. Lyngby, Denmark
Email: snaz@dtu.dk, sazuan@utem.edu.my

John Bagterp Jørgensen
Department of Applied Mathematics and Computer Science
Technical University of Denmark
DK-2800 Kgs. Lyngby, Denmark
Email: jbjo@dtu.dk

***Abstract*—Quadruple tank process is a non-linear system, have multiple manipulated and controlled variables and have significant cross binding parameters. Furthermore, the modified system is affected by some unknown measurement noise and stochastic disturbance variables which make it more complicated to model and control. In this paper, a modified quadruple-tank system has been described, all the important variables has been outlined and a mathematical model has been presented. We developed deterministic and stochastic models using differential equations and simulate the models using Matlab. Subsequently, steady state analysis is included to determine the operating window for the set points. The purpose to have an operating window for the system is to distinguish the range of feasible region to select the set points for optimum operations. Therefore, in this paper a virtual process plant is created, we investigate the operating window and construct the model in an appropriate form for future controller design.**

***Index Terms* - Modified quadruple-tank process, modeling, simulations, stochastic, steady states.**

## I. Introduction

Most of the industrial control tasks deals with systems which are mostly non-linear, have multiple inputs and outputs with complicated interactions between these manipulated and controlled variables and significant uncertainties [1]. These complicated interactions make the modified quadruple-tank system a good example to demonstrate the modeling and controller benchmarking study as discussed in [2], [3], [4].

The main purpose of this process plant is to measure and control the levels of liquid in the tanks to some desired set points. It has immeasurable disturbance variables, significant cross binding parameters which cause unwanted output disturbance when defining the control input in order to have desired output and needs linearization due to its non-linearity, which causes further errors into the control loop [5]. Moreover, due to some modification, it is also affected by unknown measurement noise and disturbance variables that are considered stochastic [6]. Therefore, it is important to have profound understanding of the underlying dynamic behaviour of the system and its potentials before implying any control strategies. The modeling of the system with different approaches have been extensively described in [7], [8], [9] whereas here it is described by deterministic and stochastic nonlinear model.

The objective of this paper is to model the modified system and to specify the model in a form appropriate for computational operation and analysis. The quadruple-tank system is based on [10] and for the modified part, we include disturbances in the upper tanks to represents the stochasticity. While [3], [4], [7] has the opportunity to work with the actual pilot plant, this work is done in a simulation environment by designing a virtual process plant. Without an actual process plant, an accurate first-principles model can be achieved by describing the non-linear dynamics of system which the equations are derived from the fundamental physical processes [11]. The conservation of mass is applied to develop simple first-principle models and we simulate the system which described by Ordinary Differential Equations (ODEs) in Matlab [12]. It is a non-stiff ODE system as all processes occurring on the same time-scale.

We elongate the study and analysis around the steady state to acquire its comportment and to develop the operating window which is sets of set points selection boundaries. These operating windows gives some basic ideas and guideline on choosing the appropriate set points in certain specific conditions of operations.

The outline of this paper is as follows. Details of the process system description will be discussed and the deterministic and stochastic model will be presented in Section II followed by the results and discussion of the simulations in Section III. It is shown that the selection of set-points uniquely determine the operating window of the system. Lastly, this paper is concluded in the final section IV.

## II. Mathematical Model

In this section we describe the modified quadruple tank system with all the important parameters are selected and variables are defined. Next we develop a deterministic model as well as a stochastic model through the use of ODEs. Then we consider the operating window for the feasible set point region through steady state analysis.

### A. System Description

The quadruple tank system comprises of four identical tanks with an outflow at the bottom, a large basin at the lower end and these tanks are connected via pipes and pumping systems, as shown in Fig.1.

The pumping system directs a fixed fraction of $F_1$ and $F_2$, denoted as $\gamma_1$ that distribute the water to Tank 1 and Tank 4, and $\gamma_2$ for Tank 2 and Tank 3 at a rate of $q_{i,in}$ $i \in \{1, 2, 3, 4\}$ respectively [13]. The values for $\gamma_1$ and $\gamma_2$

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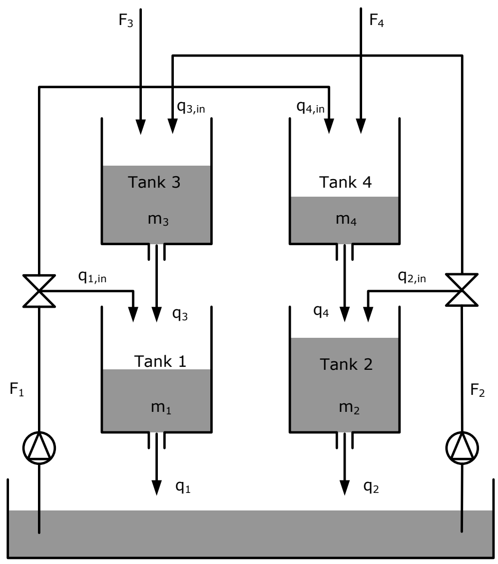


Fig. 1: Schematic diagram of the modified quadruple tank process

differs for minimum phase (RHP) and non-minimum phase (LHP). The sensors measures the height of water level in each tank, $h_i$,$i \in \{1, 2, 3, 4\}$ thus, the measured variable is affected by the noise from these sensors and due to this condition, the measured variables, $y$ consists of the actual values of the height and sensor noise. However, for the modelling purposes, we assumed no noise occurrence and impeccable measurements.

The main purpose of this process is to control the water level in Tank 1 and Tank 2 to desired set points, therefore $h_1$ and $h_2$ is selected to be the controlled variable (CVs), $z$. Usually, the controlled variables is a subset of the measured variable, $y$.

The states $x$ of the modified quadruple tank system are the masses of water in different tanks as the states of the masses changes over time due to the dynamics of the water flow in and out of each tank. Liquid is added to Tank 3 and Tank 4 resembling external disturbances which is stochastic normally distributed. The disturbances $d$ are the two unmeasured flows of $F_3$ and $F_4$ for Tank 3 and Tank 4 accordingly.

### B. Mass Balances

The non-linear model of the modified quadruple tank system is based on mass balances for each tank and the differential equations is formulated as

$$\frac{d}{dt}m_1(t) = \rho(q_{1,in}(t) + q_3(t) - q_1(t)) \tag{1a}$$
$$\frac{d}{dt}m_2(t) = \rho(q_{2,in}(t) + q_4(t) - q_2(t)) \tag{1b}$$
$$\frac{d}{dt}m_3(t) = \rho(q_{3,in}(t) + F_3(t) - q_3(t)) \tag{1c}$$
$$\frac{d}{dt}m_4(t) = \rho(q_{4,in}(t) + F_4(t) - q_4(t)) \tag{1d}$$

TABLE I: Selection of $\gamma_i$ for different operating points

| | **RHP** | **LHP** |
|---|---|---|
| $\gamma_1$ | 0.45 | 0.65 |
| $\gamma_2$ | 0.40 | 0.55 |

The mass balances (1) constitute the differential equations in the model describing the states of the system. The initial values for the mass of water in each tanks at time $t_0$ are

$$m_i(t_0) = m_{i,0} \qquad i = 1, 2, 3, 4 \tag{2}$$

### C. Inflows

Static mass balances of the two valves is used to obtain the flow rates from the valves into each of the four tanks.

$$q_{1,in}(t) = \gamma_1 F_1(t) \tag{3a}$$
$$q_{2,in}(t) = \gamma_2 F_2(t) \tag{3b}$$
$$q_{3,in}(t) = (1 - \gamma_2)F_2(t) \tag{3c}$$
$$q_{4,in}(t) = (1 - \gamma_1)F_1(t) \tag{3d}$$

$\gamma_1$ and $\gamma_2$ are constants specifying the fixed fraction of water flow with two operating points, as shown in Table I.

### D. Outflows

The height of the liquid and the flow rates out of each tanks is calculated in this section. Let the measurements and outputs be the heights, $h_i$. The height is calculated by the relations of mass and volume of the water in each tank where mass $m_i$ in tank $i$ is given by

$$m_i = \rho V_i \qquad i = 1, 2, 3, 4 \tag{4}$$

where $\rho$ is the density of the fluid and let $V_i$ be the volume of the water in all the four tanks with an assumption that $A_i$ is the cross sectional area for each tank

$$V_i = A_i h_i \qquad i = 1, 2, 3, 4 \tag{5}$$

the height $h_i$ of the liquid level in tank $i$ is calculated by the relations

$$h_i = \beta_i m_i \qquad i = 1, 2, 3, 4 \tag{6}$$

with $\beta_i$ is

$$\beta_i = \frac{1}{\rho A_i} \qquad i = 1, 2, 3, 4 \tag{7}$$

The flow rates of the liquid $q_i(t)$ is calculated by applying Bernoulli's Principle to each tank which gives the following volumetric outflow rate

$$q_i(t) = a_i\sqrt{2gh_i(t)} \qquad i = 1, 2, 3, 4 \tag{8}$$

where $a_i$ is the cross section of the pipes, $g$ is the gravity, $h_i$ is as given as (6) and $\alpha_i$ is

$$\alpha_i = a_i\sqrt{2g\beta_i} \qquad i = 1, 2, 3, 4 \tag{9}$$

the volumetric flow rate in the outlet pipes from the tanks are

$$q_i(t) = \alpha_i\sqrt{m_i} \qquad i = 1, 2, 3, 4 \tag{10}$$

All the parameter values of the modified quadruple tanks system is shown in Table II.

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TABLE II: Parameter Values

| Par | Value | Unit | Par | Value | Unit |
|---|---|---|---|---|---|
| $A_i$ | 380.1327 | $cm^2$ | $a_i$ | 1.2272 | $cm^2$ |
| $g$ | 981 | $cm/s^2$ | $\rho$ | 1.00 | $g/cm^3$ |
| $\beta_i$ | 0.0026 | $cm^2/g$ | $\alpha_i$ | 2.788 | $cm^3/sg^{(1/2)}$ |

### E. Deterministic Non-linear Model

Let $x$ indicates the state variables, $y$ is the measured variables, $u$ indicates the manipulated variables (MVs), $z$ is the controlled variables (CVs) and $d$ is the disturbances. This can be written as

$$\mathbf{x} = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \end{bmatrix}^T \tag{11a}$$

$$\mathbf{y} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 \end{bmatrix}^T \tag{11b}$$

$$\mathbf{u} = \begin{bmatrix} F_1 & F_2 \end{bmatrix}^T \tag{11c}$$

$$\mathbf{d} = \begin{bmatrix} F_3 & F_4 \end{bmatrix}^T \tag{11d}$$

$$\mathbf{z} = \begin{bmatrix} h_1 & h_2 \end{bmatrix}^T \tag{11e}$$

The measurement from the sensor is modelled as linear function, this sensor function is defined in the form of

$$y(t) = g(x(t)) \tag{12}$$

and output function is defined as

$$z(t) = h(x(t)) \tag{13}$$

By referring to (1), (3) and (10) the complete deterministic non-linear model is represented by the differential equation in the form of

$$\frac{dx(t)}{dt} = f(x(t), u(t), d(t), p) \qquad t \in [t_0, t_f] \tag{14}$$

where the parameter vector $p$ is defined as

$$p = \begin{bmatrix} a_i|_{i=0}^4 & A_i|_{i=0}^4 & \gamma_1 & \gamma_2 & g & \rho \end{bmatrix}^T \tag{15}$$

### F. Stochastic Non-linear Model

In practice, the system can be affected by noise and disturbance, therefore it is necessary to consider them in the model. In this section both process noise and measurement noise is included in the model of the system, (14) and measured variables, (12) respectively.

Consider the stochastic system

$$d\mathbf{x}(t) = f(\mathbf{x}(t), u(t), d(t), p)dt + \sigma d\mathbf{w}(t) \tag{16}$$

where $\mathbf{w}(t)$ is the process noise that is stochastic and assumed to be normally distributed, $\mathbf{w}(t) \sim N(0, R_w)$, due to the unknown information regarding the distribution.

Let the measurement noise, $\mathbf{v}(t)$ signifies the occurrence of noise from the sensors during the process of measuring in each tank and it is normally distributed, $\mathbf{v}(t) \sim N(0, R_v)$. $\mathbf{v}(t)$ is being added to the measured variables, (12) giving

$$\mathbf{y}(t) = g(\mathbf{x}(t)) + \mathbf{v}(t) \tag{17}$$

In this work, let the input flow rates be the sum of the deterministic component and the stochastic component,

$$\begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} F_{1s} \\ F_{2s} \end{bmatrix} + \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} \sim \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 12.5^2 & 0 \\ 0 & 12.5^2 \end{bmatrix} \right) \tag{19}$$

and let all sensors be independent and measure the level in each and every tanks. The measurement noise for all sensors is assumed to have the same variance,

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \\ \mathbf{y}_4 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} + \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{bmatrix}, \tag{20}$$

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{bmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2^2 & 0 & 0 & 0 \\ 0 & 2^2 & 0 & 0 \\ 0 & 0 & 2^2 & 0 \\ 0 & 0 & 0 & 2^2 \end{bmatrix} \right) \tag{21}$$

The deterministic and stochastic non-linear model is presented. The results of the simulation will be presented in the next section.

### G. Steady State Analysis

At steady state, the outflows and inflows of each tank are identical. Therefore, (1) in combination with (10) can be combined into

$$q_{1,s} = q_{1,in,s} + q_{3,s} = \gamma_1 F_{1,s} + (1-\gamma_2)F_{2,s} + F_{3,s} \tag{22a}$$

$$q_{2,s} = q_{2,in,s} + q_{4,s} = \gamma_2 F_{2,s} + (1-\gamma_1)F_{1,s} + F_{4,s} \tag{22b}$$

$$q_{3,s} = q_{3,in,s} + F_{3,s} = (1-\gamma_2)F_{2,s} + F_{3,s} \tag{22c}$$

$$q_{4,s} = q_{4,in,s} + F_{4,s} = (1-\gamma_1)F_{1,s} + F_{4,s} \tag{22d}$$

$$m_{1,s} = \left( \frac{\gamma_1 F_{1,s} + (1-\gamma_2)F_{2,s} + F_{3,s}}{\alpha_1} \right)^2 \tag{23a}$$

$$m_{2,s} = \left( \frac{\gamma_2 F_{2,s} + (1-\gamma_1)F_{1,s} + F_{4,s}}{\alpha_2} \right)^2 \tag{23b}$$

$$m_{3,s} = \left( \frac{(1-\gamma_2)F_{2,s} + F_{3,s}}{\alpha_3} \right)^2 \tag{23c}$$

$$m_{4,s} = \left( \frac{(1-\gamma_1)F_{1,s} + F_{4,s}}{\alpha_4} \right)^2 \tag{23d}$$

$$h_{i,s} = \beta_i m_{i,s} \qquad i = 1, 2, 3, 4 \tag{24}$$

If the liquid heights in Tank 1 and Tank 2, $h_{1,s}$ and $h_{2,s}$, are specified, the corresponding masses, $m_{1,s} = h_{1s}/\beta_1$ and $m_{2s} = h_{2s}/\beta_2$, would also be specified and so would the outflow rates from Tank 1 and Tank 2, $q_{1,s} = \alpha_1 \sqrt{m_{1,s}}$ and $q_{2,s} = \alpha_2 \sqrt{m_{2,s}}$. Consequently, the required steady state manipulable flow rates, $F_{1,s}$ and $F_{2,s}$, must satisfy

$$\underbrace{\begin{bmatrix} \gamma_1 & 1-\gamma_2 \\ 1-\gamma_1 & \gamma_2 \end{bmatrix}}_{=M} \begin{bmatrix} F_{1,s} \\ F_{2,s} \end{bmatrix} = \begin{bmatrix} q_{1,s} - F_{3,s} \\ q_{2,s} - F_{4,s} \end{bmatrix} \tag{25}$$

The coefficient matrix, $M$, is singular when

$$\det M = \gamma_1\gamma_2 - (1-\gamma_1)(1-\gamma_2) = 0 \tag{26}$$

i.e when

$$\gamma_2 = 1 - \gamma_1 \tag{27}$$

and (25) does not have a unique solution. When $\det M \neq 0$, (25) has a unique solution given by

$$\begin{bmatrix} F_{1,s} \\ F_{2,s} \end{bmatrix} = \begin{bmatrix} \gamma_1 & 1-\gamma_2 \\ 1-\gamma_1 & \gamma_2 \end{bmatrix}^{-1} \begin{bmatrix} q_{1,s} - F_{3,s} \\ q_{2,s} - F_{4,s} \end{bmatrix} \tag{28}$$

For the case when $\det M = 0$, an extra condition, e.g. $F_{1,s} = F_{2,s}$, can be used to obtain a unique solution. In the case $F_{1,s} = F_{2,s}$, the unique solution is given by

$$F_{1,s} = F_{2,s} = \frac{q_{1,s} - F_{3,s}}{\gamma_1 + (1-\gamma_2)} = \frac{q_{2,s} - F_{4,s}}{(1-\gamma_1) + \gamma_2} \tag{29}$$

if it exists. The solution only exists if the latter equality in (29) is satisfied. To solve (28), temporarily let $b_1 = (q_{1,s} - F_{3,s})$ and $b_2 = q_{2,s} - F_{4,s}$

$$\begin{bmatrix} F_{1,s} \\ F_{2,s} \end{bmatrix} = \frac{1}{detM} \begin{bmatrix} \gamma_2 b_1 - (1-\gamma_2) b_2 \\ \gamma_1 b_2 - (1-\gamma_1) b_1 \end{bmatrix} \tag{30}$$

With the manipulation of (30) the equation can be transform into two condition,

*Condition 1: det M > 0*

$$\frac{1-\gamma_1}{\gamma_1} \leq \frac{q_{2,s} - F_{4,s}}{q_{1,s} - F_{4,s}} \leq \frac{\gamma_2}{1-\gamma_2} \tag{31}$$

*Condition 2: det M < 0*

$$\frac{\gamma_2}{1-\gamma_2} \leq \frac{q_{2,s} - F_{4,s}}{q_{1,s} - F_{4,s}} \leq \frac{1-\gamma_1}{\gamma_1} \tag{32}$$

By inserting (24) into (10) and $\eta_i$ is given by,

$$\eta_i = \frac{\alpha_i}{\sqrt{\beta_i}} \qquad i = 1,2 \tag{33}$$

the possible feasible region of set points of the heights in Tank 2, $h_{2,s}$ can de determined by

$$h_{2,lb} \leq h_{2,s} \leq h_{2,ub} \tag{34}$$

where $h_{2,lb}$ and $h_{2,up}$ is upper and lower bound of $h_2$ respectively with

$$h_{2,lb} = \left[ \frac{\left(\frac{1-\gamma_1}{\gamma_1}\right)\left(\eta_1\sqrt{h}_{1,s} - F_{3,s}\right) + F_{4,s}}{\eta_2} \right]^2$$

$$h_{2,ub} = \left[ \frac{\left(\frac{\gamma_2}{1-\gamma_2}\right)\left(\eta_1\sqrt{h}_{1,s} - F_{3,s}\right) + F_{4,s}}{\eta_2} \right]^2$$

This corresponds to solving

$$f(x_s, u_s, d_s, p) = 0 \tag{35a}$$

$$y_s = g(x_s, p) \tag{35b}$$

$$z_s = h(x_s) \tag{35c}$$

for $x_s$ when $u_s$, $d_s$ and $p$ are given.

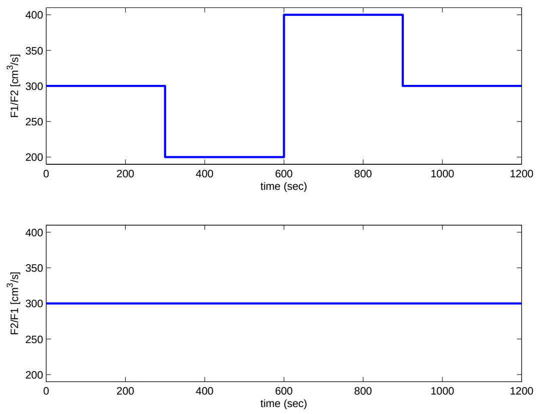


Fig. 2: Multiple steps input for MVs $u_1$ and $u_2$

## III. Simulation of the System

Now that the deterministic and stochastic model is obtained, we visualized the behaviour of the system and verify the model by simulations. To have the basic understanding of the operation and the directions of the water flow, step test response is applied and observed. To identify the model, multiple step experiments with different step sizes and level of noise for all input/output combinations were carried out.

### A. Step Responses

The first part of the experiments is implemented to have an overview of the water flow in the connected tanks by changing the step inputs. Starts with applying multiple steps to the first manipulated variable, $F_1$ and then followed by the second one, $F_2$ to the deterministic model, one at a time as shown in Fig.2. For these tests, $\gamma_i$ is chosen to be on the RHP.

The response is exhibited in Fig.3 for $u_1$ set point changes and Fig.4 for $u_2$ set point changes. The system reacts to the changes quite rapidly and illustrates the dynamic of the water flow between the connected tanks. Changes in $F_1$ results in deviations of mass of $m_1$, $m_2$ and $m_4$ which represents the operation of the first pumping system. The operation of the second pumping system can be seen in Fig. 4 where changes in $F_2$ does not give any effects to the mass of $m_4$ but we can see the changes of mass accordingly in the other tanks.

### B. Transfer Function Identification

Fig. 5 shows the normalized single step response for 10%, 25% and 50% increases in $F_1$ and $F_2$ respectively for the deterministic model without measurement noise. The differences between the step increases are insignificant but existent and this is expected for the non-linear model, whereas the responses would be identical if the model is linear. The transfer function, $G$ is estimated from these normalized step responses from $U$ to $Y$ ,

$$Y = GU \tag{36}$$

From Fig.5 the transfer function for $h_1$ and $h_4$ is first order, for $h_2$ is second order and zero for $h_3$. Subsequently, with input of $F_2$ it can be seen that the transfer function for $h_1$ is

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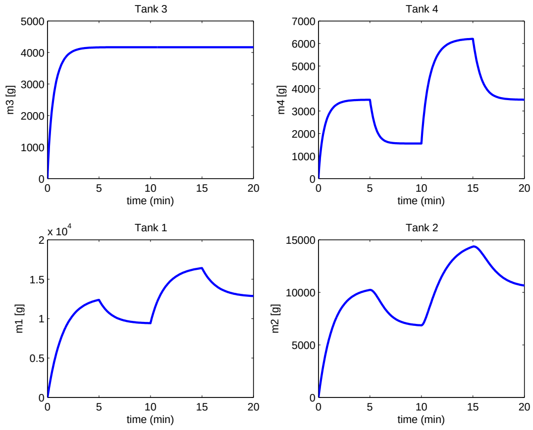


Fig. 3: Responses of mass in Tank 1, 2 and 4 due to $u_1$ set point changes.

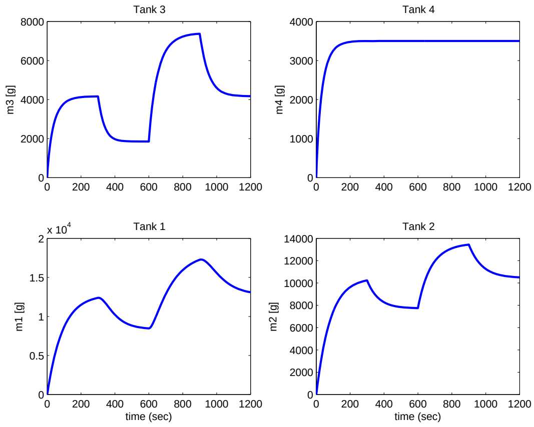


Fig. 4: Responses of mass in Tank 1, 2 and 3 due to $u_2$ set point changes

second order, for $h_2$ and $h_3$ is first order and zero for $h_4$. The transfer functions identified is as follows:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \\ G_{31} & G_{32} \\ G_{41} & G_{42} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{37}$$

$$G_{11} = \frac{0.11}{130.2s+1} \qquad G_{12} = \frac{0.14}{(1185s^2+171s+1)}$$

$$G_{21} = \frac{0.12}{(15311.3s^2+145.2s+1)} \qquad G_{22} = \frac{0.08}{148.2s+1}$$

$$G_{31} = 0 \qquad G_{32} = \frac{0.09}{111.7s+1}$$

$$G_{41} = \frac{0.08}{113.5s+1} \qquad G_{42} = 0$$

In the next simulation, to identify the noise, we include noise to visualize the step responses with measurement noise. Fig.6 shows three different step sizes (10%, 25% and 50%) normalized step responses in $F_1$ and $F_2$ respectively with low, medium and high noise. From the transfer function, the noise is estimated with an assumption that

$$Y = GU + E \tag{38}$$

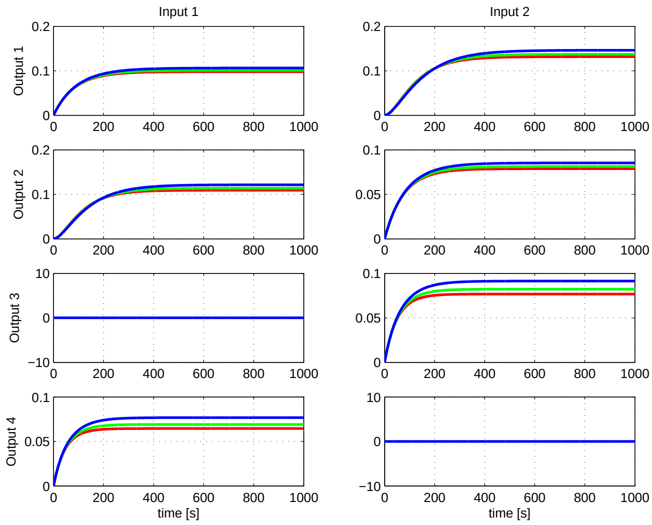


Fig. 5: Normalized step response for 10%, 25% and 50% increases in $F_1$ and $F_2$

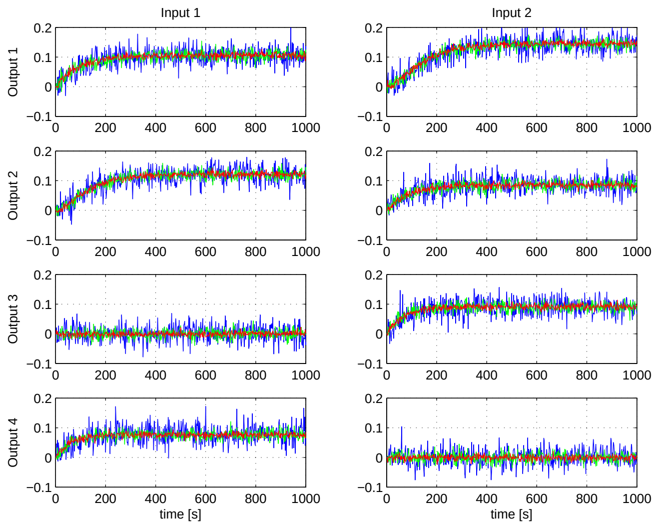


Fig. 6: 10%, 25% and 50% normalized step increases in $F_1$ and $F_2$ with noise

where $G$ is the transfer function from input $U$ to output $Y$ and $E$ is the noise. It is estimated as

$$E = Y - GU \tag{39}$$

where $Y$ is the noisy measurement.

Fig. 6 shows the simulation results of the stochastic part and it can be seen that the process noise, $\mathbf{w}_t$ is assumed to be constant.

### C. Operating Window in Steady State

In previous section, we derived and investigated the steady state to determine the possible feasible region of set points and suitable operating window for the modified quadruple tank system. It is non trivial decision to choose the appropriate

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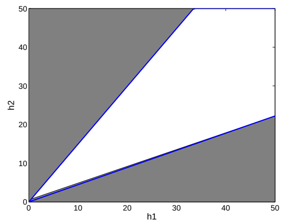
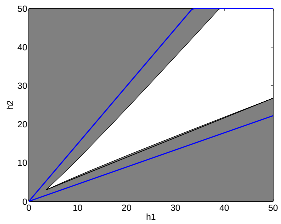
range of set point, $h_2$ for certain conditions such as the value of $\gamma_i$ and disturbances $F_3$ and $F_4$ with min and max bound of disturbances selections is $0 \leq d \leq 100$. The simulation of the analysis shows the possible region of set point selections.

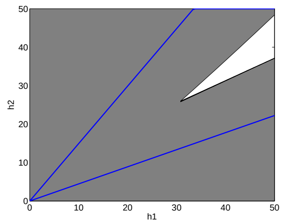
The white region in Fig. 7 and 8 shows the operating window that is feasible for different disturbance conditions. Referring to Fig. 7a and 8a, although the feasible region to choose $h_2$ is wide, it can be seen that it is bounded with upper and lower limits, and if we refer to Fig. 7c and 8c it is almost not possible to choose the middle set points. Comparing Fig.7d and Fig.8d, with different selection of the fixed fraction of the water flow $\gamma_i$, the modified quadruple tank system is able to operate at the maximum value of disturbances during LHP operation.



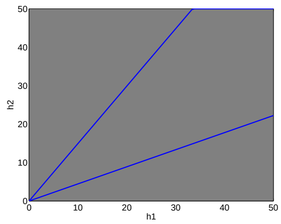

(a) With $F_3 = F_4 = 0$




(b) With $F_3 = F_4 = 25$




(c) With $F_3 = F_4 = 75$




(d) With $F_3 = F_4 = 100$

Fig. 7: Operating window with $\gamma_i$ in the RHP

## IV. Conclusion

In this paper the modified quadruple tank system is presented, a virtual plant of the system is created and the dynamics of the system is modelled. The model comprises of deterministic non-linear model and stochastic non-linear model is simulated and then the steady state analysis is investigated in order to obtained an operating window for the set points. The verdict is not all set points combinations are possible for the system to attain its desirable outcomes. Therefore this operating window is not negligible and useful for future controller design. Besides, the non-linear model is reform prior to future implementation of computer control system and analysis.

## V. Acknowledgement

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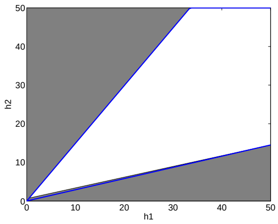
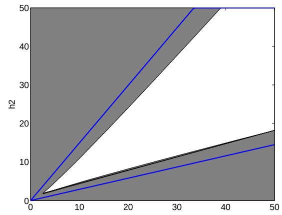
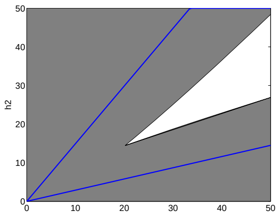


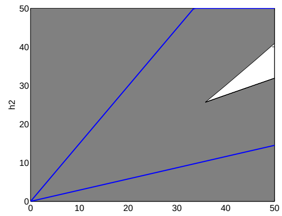

(a) With $F_3 = F_4 = 0$




(b) With $F_3 = F_4 = 25$




(c) With $F_3 = F_4 = 75$




(d) With $F_3 = F_4 = 100$

Fig. 8: Operating window with $\gamma_i$ in the LHP

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# PAPER B

# Linear discrete-time state space realization of a modified quadruple tank system with state estimation using Kalman Filter

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Sazuan Nazrah Mohd. Azam

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# Linear Discrete-time State Space Realization of a Modified Quadruple Tank System with State Estimation using Kalman Filter

**Sazuan N M Azam**

Department of Applied Mathematics and Computer Science, Technical University of Denmark, DK-2800 Kgs. Lyngby, Denmark

E-mail: snaz@dtu.dk

**Abstract.** In this paper, we used the modified quadruple tank system that represents a multi-input-multi-output (MIMO) system as an example to present the realization of a linear discrete-time state space model and to obtain the state estimation using Kalman filter in a methodical mannered. First, an existing dynamics of the system of stochastic differential equations is linearized to produce the deterministic-stochastic linear transfer function. Then the linear transfer function is discretized to produce a linear discrete-time state space model that has a deterministic and a stochastic component. The filtered part of the Kalman filter is used to estimates the current state, based on the model and the measurements. The static and dynamic Kalman filter is compared and all results is demonstrated through simulations.

## 1. Introduction

A dynamic model of a system can be described in a various way. From the derivation of the mathematical model it is possible to obtain the underlying information of the system and implement a control algorithm to the system. Most of the systems or processes are usually described by state-space system and by investigating the state of a system at certain time and its present and future inputs, it is possible to predict the output in the future [1]. State space models can be either non-linear or linear form and usually a real system or process is described by a non-linear models whereas in order to estimate and control the system, most mathematical tools are more accessible to a linear models. Therefore, in this paper we want to demonstrate the transformation of a non-linear continuous model of a modified quadruple tank system described as deterministic-stochastic differential equations into a linear discrete-time state space model.

Throughout this work, we will fully utilize the Modified Quadruple Tank System, based on [2] to assimilate the fundamental theory of model realization and state estimation to an exemplification of MIMO system, illustration of the real-world complex system applications which is widely used for education in modeling and demonstrating advanced control strategies [3], [4].

Several works have been done on four tanks system regarding the modeling the dynamic of the system. A full description of linearization of the model for four tank system is presented in [5], [6] and [7]. In [7] the linearization is described in detail using the Jacobian matrix formation to represent the system in state space model while in [8] establishes the linearized model based on the non-linear mechanism of the system. Another method is shown in [9]where the model is

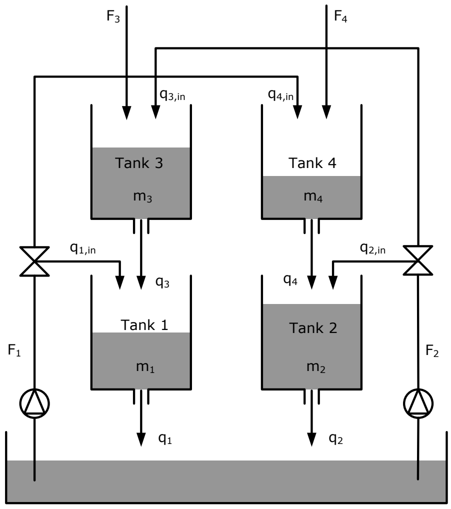


**Figure 1.** Schematic diagram of the modified quadruple tank process

developed from input and output data resulting an empirical linear state space model using a sub-space identification and can be used effectively for a non-linear system.

As for the state estimation, we want to estimates all the variables which represents the internal condition or the status of the system at a specific given time [1] so as to allow for future output prediction and to design the control algorithms. A Kalman's state estimator for a non-linear multivariable process such as the four tanks system is shown in [10] using a linear state space model solved by the algebraic Ricatti equation. Meanwhile the usage of the estimation of the Kalman filter with full derivation can also be found in [11]. In this work we use the Kalman filter in order to estimates the current state of the modified quadruple tank system and evaluate the response of dynamic and static Kalman filter.

This paper is structured as follows. A brief description of modified quadruple tank system is presented and the realization of the linear discrete-time state space model is shown in detail in Section 2. Then the state estimation using Kalman filter is discussed in Section 3. The following section is where all the results is discussed in Section 4. Finally, we conclude this work in the last section.

## 2. Linear Discrete-time State Space Model Realization

The first part of this paper is to transform the non-linear continuous state space model of a modified quadruple tank system to a linear discrete state space model through linearization and discretization. A brief description of the system is presented below and followed by the linearization and discretization.

### *2.1. The Modified Quadruple Tank System*

The modified quadruple tank system is a simple process, consist of four identical tanks and two pumping system as shown in Figure 1 but yet illustrates a system that is non-linear with multiple inputs and outputs (MIMO) and complicated interactions between manipulated and controlled variables.

The main objective of this system is to control the level of the water in the lower tanks (Tank 1 and 2) by manipulating the flow rates $F_1$ and $F_2$ which are distributed across all four tanks, represents the dynamics of multivariable interaction since each manipulated variables influences the outputs. The height of the water level in these two tanks, $h_1$ and $h_2$ is measured and controlled. The flows denoted $F_3$ and $F_4$ are unmeasured unknown disturbances.

The dynamic of the process is described in Stochastic Nonlinear Model (SDE) given as:

$$d\mathbf{x}(t) = f(\mathbf{x}(t), u(t), d(t), p)dt + \sigma d\mathbf{w}(t) \tag{1a}$$

$$\mathbf{y}(t) = c(\mathbf{x}(t)) + \mathbf{v}(t) \tag{1b}$$

$$\mathbf{z}(t) = c(\mathbf{x}(t)) \tag{1c}$$

where $\mathbf{w}(t)$ is the process noise and normally distributed, $\mathbf{w}(t) \sim N(0, R_w)$, due to the unknown information regarding the distribution and $\mathbf{v}(t)$ is the measurement noise from the sensors in each tank and it is normally distributed, $\mathbf{v}(t) \sim N(0, R_v)$. $\mathbf{v}(t)$ is being added to the measured variables, (1b). For full description of the modeling part, see [12].

### *2.2. Linear System Realization*

Linearization is required to find the linear approximation to analyze the behaviour of the nonlinear function, given a desired operating point. We apply the first-order term only of Taylor expansion by truncation around the steady state of the non-linear differential equations, $f(x(t), u(t), d(t))$ and consider the derivative of the state variable, $x$. This derivative is defined as a function, $f$,

$$f(x(t), u(t), d(t), p) = \begin{pmatrix} \rho\gamma_1 u_1(t) + \alpha_3 - \alpha_1 \\ \rho\gamma_2 u_2(t) + \alpha_4 - \alpha_2 \\ \rho(1-\gamma_2)u_2(t) + \rho d_1(t) - \alpha_3 \\ \rho(1-\gamma_1)u_1(t) + \rho d_2(t) - \alpha_4 \end{pmatrix} \tag{2}$$

where $\alpha_i$ is given by

$$\alpha_i = \rho a_i \sqrt{x_i(t)} \sqrt{\frac{2g}{\rho A_i}} \qquad i = 1, 2, 3, 4 \tag{3}$$

and $p$ denote the vector containing all the parameters of the system, for full description of the parameter see [12]. The Jacobian of $f$ with respect to the state-variables are

$$J_x(x(t), u(t), d(t), p) = \begin{pmatrix} -\beta_1 & 0 & \beta_3 & 0 \\ 0 & -\beta_2 & 0 & \beta_4 \\ 0 & 0 & -\beta_3 & 0 \\ 0 & 0 & 0 & -\beta_4 \end{pmatrix} \tag{4}$$

where $\beta_i$ is given by

$$\beta_i = \frac{1}{\sqrt{x_i(t)}} \sqrt{\frac{a_i^2 g \rho}{2A_i}} \qquad i = 1, 2, 3, 4 \tag{5}$$

Similarly the Jacobian of $f$ with respect to the manipulated variables giving

$$J_u(x(t), u(t), d(t), p) = \begin{pmatrix} \rho\gamma_1 & 0 \\ 0 & \rho\gamma_2 \\ 0 & \rho(1-\gamma_2) \\ \rho(1-\gamma_1) & 0 \end{pmatrix} \tag{6}$$

and lastly the Jacobian of $f$ with respect to the disturbance variables are

$$J_d(x(t), u(t), d(t), p) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \rho & 0 \\ 0 & \rho \end{pmatrix} \tag{7}$$

In this case, we introduced the deviation variables as

$$X(t) = x(t) - x_s \quad U(t) = u(t) - u_s \quad D(t) = d(t) - d_s \tag{8}$$

and defined the Jacobian matrices evaluated around a stationary point $x_s, u_s, d_s$ to be

$$A_c = J_x(x_s, u_s, d_s, p) \quad B_c = J_u(x_s, u_s, d_s, p) \quad E_c = J_d(x_s, u_s, d_s, p) \tag{9}$$

With these matrices the first order Taylor approximation around the steady state point are given as

$$\begin{aligned} f(x(t), u(t), d(t), p) &\approx f(x_s, u_s, d_s, p) + A_c X(t) \\ &+ B_c X(t) + E_c D(t) \\ &= A_c X(t) + B_c X(t) + E_c D(t) \end{aligned} \tag{10}$$

For the measurement and controlled variables we introduced $Y(t)$ and $Z(t)$ respectively and the linearized system of the modified quadruple tank system as

$$\dot{X}(t) = A_c X(t) + B_c X(t) + E_c D(t) \qquad X(t_0) = 0 \tag{11a}$$

$$Y(t) = CX(t) \tag{11b}$$

$$Z(t) = C_z X(t) \tag{11c}$$

where the $C$ matrices are defined as

$$C = C_z = \begin{pmatrix} \frac{1}{\rho A_1} & 0 & 0 & 0 \\ 0 & \frac{1}{\rho A_2} & 0 & 0 \end{pmatrix} \tag{12}$$

### *2.3. Discretization of a Linear System*

The dynamics of the modified quadruple tank system is now described as (11) and to use this linear continuous model of the system to be subjected to MPC, the model needs to be discretized by assuming zero-order-hold (ZOH) of the variables at specified sampling points, that is assuming the exogenous variables are constant between sampling points. The aim is to have a linear discrete-time state space model with piecewise constant $u_k$, $d_k$ in a form of

$$x_{k+1} = A_d x_k + B_d u_k + E_d d_k \tag{13a}$$

$$y_k = C_d x_k + D_d u_k \tag{13b}$$

with discrete-time consideration

$$\begin{aligned} t_k &= t_0 + kTs, \qquad k = 0, 1, 2... \\ x_k &= x(t_k) \end{aligned}$$

and assuming the inputs on the ZOH is

$$u(t) = u_k, \qquad t_k \le t \le t_{k+1}$$

then the solution of (13) with respect to $u$ is given as

$$x_{k+1} = x(t_{k+1}) \tag{14a}$$

$$= e^{A(t_{k+1} - t_k)} x_k + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1} - \tau)} B u(\tau) d\tau \tag{14b}$$

$$= \left[e^{AT_s}\right] x_k + \left[\int_0^{T_s} e^{A\eta} B d\eta\right] u_k \tag{14c}$$

By comparing both equations (11) and (14) and similar result can be obtained for disturbances variable $d(t)$ giving

$$\begin{array}{lll} A_d = e^{AT_s} & B_d = \int_0^{T_s} e^{A\tau} B d\tau & E_d = \int_0^{T_s} e^{A\tau} E d\tau \\ C_d = C & D_d = D & \end{array} \tag{15}$$

where $A_d, B_d, E_d$ can be computed with

$$\begin{aligned} \begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} &= \exp\left( \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} T_s \right) \\ \begin{bmatrix} A_d & E_d \\ 0 & I \end{bmatrix} &= \exp\left( \begin{bmatrix} A & E \\ 0 & I \end{bmatrix} T_s \right) \end{aligned} \tag{16}$$

For this particular work, the continuous state space representation matrices were discretized with $T_s = 30s$ assuming ZOH.

Considering the stochastic part of the model, a piecewise constant process noise $w$, measurement noise $v$ and uncertainty of the initial state $x_0$ to the process is added. The linear discrete model from (13) is expanded into stochastic version as in the equation below

$$x_{k+1} = A_d x_k + B_d u_k + E_d(d_k + w_k) \tag{17a}$$
$$y_k = C_d x_k + v_k \tag{17b}$$
$$z_k = C_{dz} x_k + v_k \tag{17c}$$

subject to

$$x_0 \sim N(\bar{x_0}, P_p), \quad w_k \sim N(0, Q), \quad v_k \sim N(0, R) \tag{18}$$

where $Q, R$ is given by

$$Q = \begin{bmatrix} 12.5^2 & 0 \\ 0 & 12.5^2 \end{bmatrix} \quad R = \begin{bmatrix} 2^2 & 0 & 0 & 0 \\ 0 & 2^2 & 0 & 0 \end{bmatrix}$$

and $P_p$ is given by

$$P_p = \begin{bmatrix} 0.1^2 & 0 & 0 & 0 \\ 0 & 0.1^2 & 0 & 0 \\ 0 & 0 & 0.1^2 & 0 \\ 0 & 0 & 0 & 0.1^2 \end{bmatrix}$$

*2.4. Linear Discrete-time State Space Representation*

In order to rewrite the difference equation system representation (13) in a more structured form, the Markov parameters is introduced. It is a discrete impulse coefficients of a discrete state space model. The Markov parameters are calculated to avoid making iterative simulations to keep only the matrix-vector multiplications. In doing so, a significant time saving is introduced to the control algorithm and to have an observer canonical form with minimal realization. Let $H_i$ denote the Markov parameters at the $i'th$ sampling time after an unit-impulse, then to obtain the Markov parameters from $u$ to $y$ is given as

$$H_i = \begin{cases} 0 & i = 0 \\ C{A_d}^{i-1} B & i = 1, 2, ...N \end{cases} \tag{19}$$

$N$ is assigned value to be sufficiently large so that the impulse response can reach the steady state. The Markov parameters for $u$ to $z$, $d$ to $y$ and $d$ to $z$ is computed the same way and by replacing the appropriate matrices accordingly. With all the information being gathered, it can be re-written in a matrix form of

$$Y = \Phi x_0 + \Gamma U \tag{20}$$

where $Y$, $\Phi$, $U$ are

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_i \end{bmatrix} \quad \Phi = \begin{bmatrix} CA_d \\ CA_d{}^2 \\ CA_d{}^3 \\ \vdots \\ CA_d{}^i \end{bmatrix} \quad U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_i \end{bmatrix}$$

while $\Gamma$ is obtained from the calculated Markov parameters, $H_i$, $i = 1, 2, ...N$

$$\Gamma = \begin{bmatrix} H_1 & 0 & 0 & \dots & 0 \\ H_2 & H_1 & 0 & \dots & 0 \\ H_3 & H_2 & H_1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ H_N & H_{N-1} & H_{N-2} & \vdots & H_1 \end{bmatrix}$$

As for the system with disturbances, the state space model can be represented as

$$Y = \Phi x_0 + \Gamma_u U + \Gamma_d D \tag{21}$$

where

$$D = \begin{bmatrix} d_1 & d_2 & d_3 & \dots & d_i \end{bmatrix}^T$$

From equations (20) and (21), $\Phi$ and $\Gamma$ can be used for the prediction part from the Kalman filter for a model predictive control strategy.

## 3. State Estimation for the Discrete-Time Linear System

From the previous section, the discrete-time state space model is a linearized model from the non-linear model. We want to extract information from the measurements of the real system in order to limit the discrepancy between the model and the real system but since the models assume measurement error, the signals need to be filtered. This can be done by using Kalman filter where it is used to filter the measurement [1]. The Kalman filter consists of two parts, filtering part and prediction part. The filtered part is to estimates current state based on the model and the measurements whilst the prediction part is used by the constrained regulator to predict the future output trajectory, given an input trajectory. This is illustrated in the block diagram as in Figure 2. In this paper we focus on the filtering part for the state estimation only and design both dynamic and static filter to evaluate their estimation.

### *3.1. Dynamic Kalman Filter*

From [12] the model is linear time invariant (LTI) discrete-time stochastic difference equations, in the form of

$$x_{k+1} = A_d x_k + B_d u_k + E_d d_k + E d w_k \tag{22a}$$
$$y_k = C_d x_k + v_k \tag{22b}$$

subject to

$$w_k \sim N(0, Q), v_k \sim (0, R) \tag{23}$$

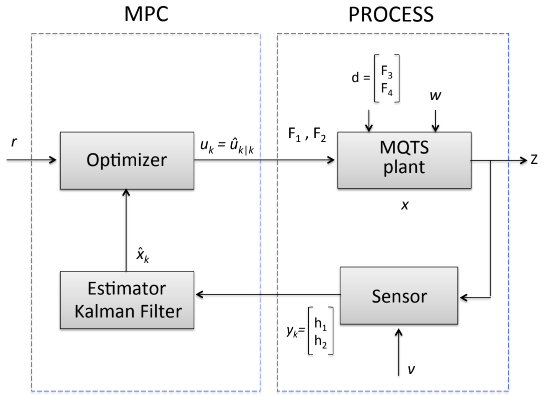


**Figure 2.** Block diagram of MPC for modified quadruple tank process

where the process noise $w_k$ and measurement noise $v_k$ are distributed as

$$\begin{bmatrix} w_k \\ v_k \end{bmatrix} \sim N_{iid}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\right) \tag{24}$$

where $R$ and $Q$ is the covariance matrix of measurement error and disturbances variable accordingly, $S$ is the covariance matrix between disturbance variable and measurement error and the distribution of the initial state is given by

$$x_{0|-1} \sim N(\hat{x}_{0|-1}, P_{0|-1}) \tag{25}$$

Assuming at stationary point $t = t_k$ and the measurement $y_k = y(t_k)$, the filtering part can be performed by calculating

$$\hat{y}_{k|k-1} = C\hat{x}_{k|k-1} \tag{26a}$$
$$e_k = y_k - \hat{y}_{k|k-1} \tag{26b}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx,k}e_k \tag{26c}$$
$$\hat{w}_{k|k} = K_{fw}e_k \tag{26d}$$
$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k + \hat{w}_{k|k} \tag{26e}$$

By using the coefficients

$$R_{e,k} = CP_{k|k-1}C^T + R \qquad K_{fx,k} = P_{k|k-1}C^T R_{e,k}^{-1} \qquad K_{fw} = SR_{e,k}^{-1} \tag{27}$$

and the following expression can be achieved

$$P_{k+1|k} = AP_{k|k}A^T + Q_{k|k} - AK_{fx,k}S^T - SK_{fx,k}^T A^T \tag{28}$$

*3.2. Static Kalman Filter*

From equation (28) it can be re-written into a difference equation form as

$$P_{k+1|k} = AP_{k|k-1}A^T + Q - (AP_{k|k-1}C^T + S)(CP_{k|k-1}C^T + R)^{-1}(AP_{k|k-1}C^T + S)^T \tag{29}$$

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$P$ signifies the stationary one-step ahead state error covariance matrix obtained from the Discrete-time Algebraic Riccati Equation (DARE).

$$P = APA^T + Q - (APC^T + S)(CPC^T + R)^{-1}(APC^T + S)^T \tag{30}$$

and the coefficients in equations (27) can be simplify

$$R_e = CPC^T + R \qquad K_{fx} = PC^T R_e^{-1} \qquad K_{fw} = SR_e^{-1} \tag{31}$$

Since by using this limit as an approximation to the one-step matrix and the Kalman gains $K_{fx}$ and $K_{fw}$ becomes constant matrices, it will lighten the computations of the controller.

## 4. Results and Simulation of the system

The first part of this work is the linear discrete-time state space realization and computations which is re-written in a more structured form where we introduced the use of Markov parameters. Then the second part is the implementation of Kalman filter and the predictive controller strategy. In this sections, all results from the computations and simulations will be shown.

### *4.1. Linear Discrete-time State Space Realization*

The linearized continuous system matrices are obtained as in equation (11) to ensure that the theoretical linear estimation of the system are almost identical to the non-linear system, it is

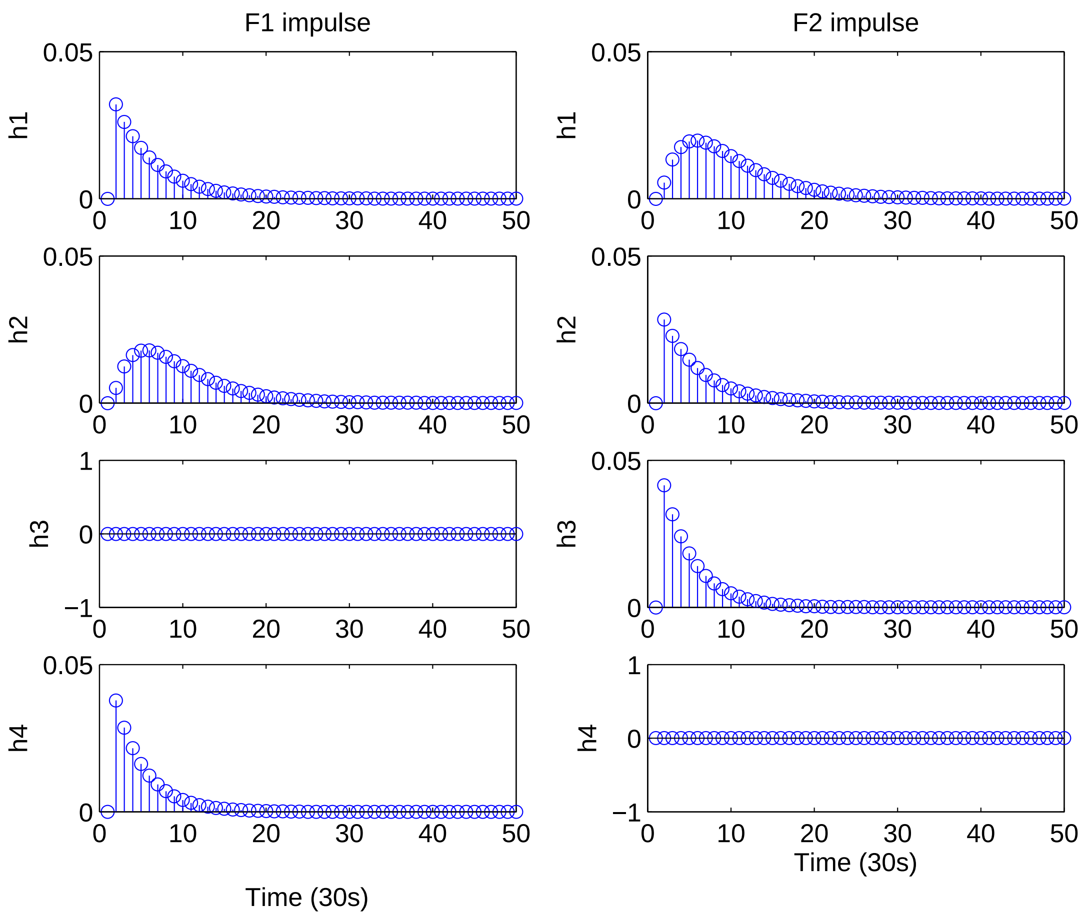


**Figure 3.** Markov Parameters for the Discrete-time State Space Model Experiments

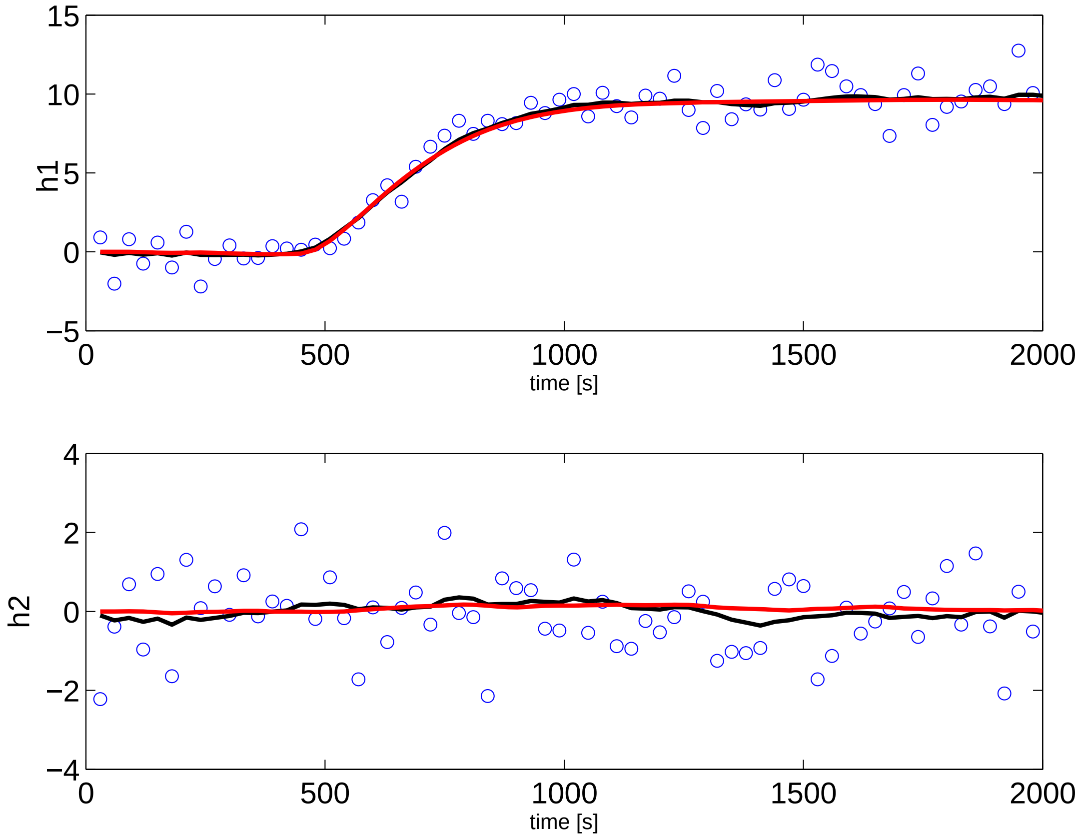


**Figure 4.** Kalman filter for noisy $F_3$ and 10% step changes

possible to compare the estimated gains and time-constants from the non linear transfer function. Then, by assuming zero-order hold, the continuous state space representation matrices above were discretized with $T_s = 30s$, hence the discrete state space system matrices is also obtained as in equation (13).

Considering a sampling time of $T_s = 30s$ the Markov parameters has been calculated for the discrete-time state space model, from Figure 3 the plots can be compared to the model from the step response experiment in our previous work [12] and shows that in most cases the plots are very similar. Therefore, the calculated Markov parameters, $H_i$ are reliable and usable for other purposes such as in designing the predictive controller.

*4.2. State Estimation*

In this experiments, both dynamic and static Kalman filter were tested as a state estimator where the disturbance is an unknown stochastic variable, then after approximately 450s we introduced a 10% step changes and for this simulations, the linear model is used to create the measurements. Figure 4 and Figure 5 were plotted to compare the estimated current states with and without Kalman filter and between dynamic and static Kalman filter with a step change of $F_3$ and $F_4$ accordingly. It can be clearly seen that in general, the filter is well performed tracking the output trajectory from the noisy measurements and also it can cope well dealing with an impact of the unknown disturbance step. Although the difference between dynamic and static Kalman filter is not apparent, the dynamic filter is able to even further reduce the noise giving a smoother and more stable response particularly in tanks $h_1$ in figure 4 and tank $h_2$ in figure 5, noticeable for both tanks that is directly affected by the given step disturbance.

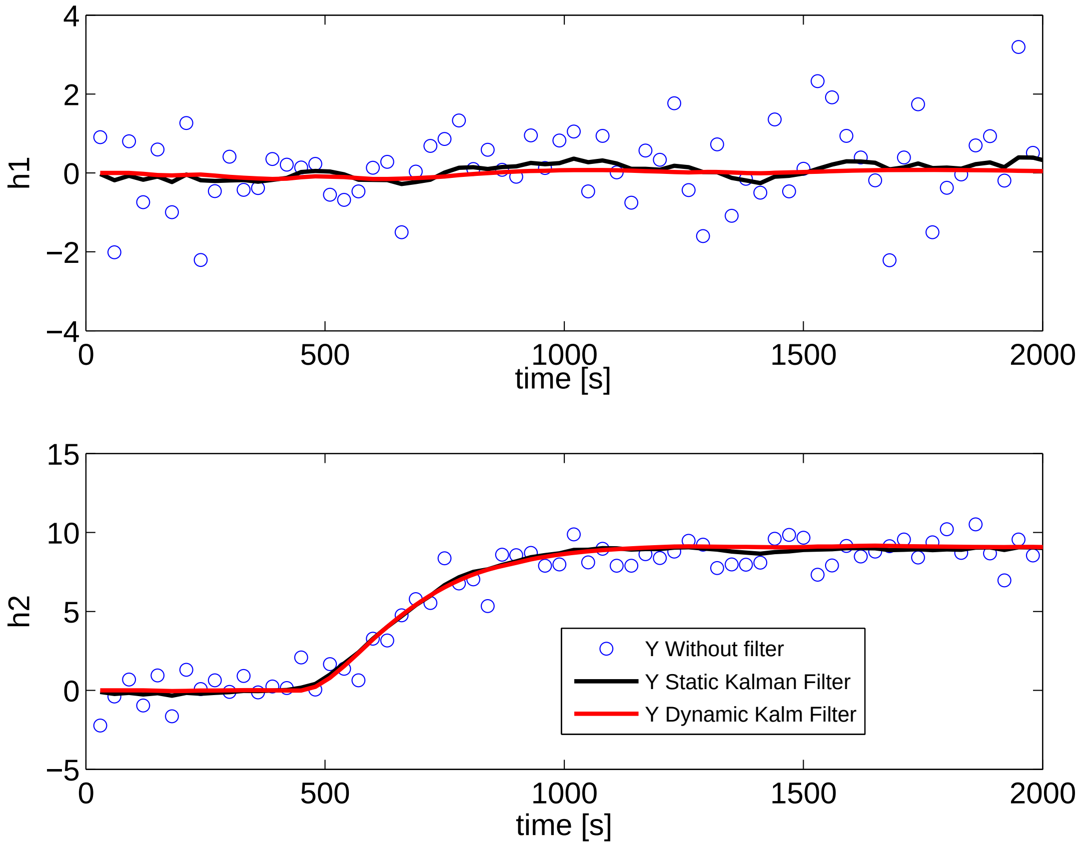


**Figure 5.** Kalman filter for noisy $F_4$ and 10% step changes

## 5. Conclusion

This paper has described comprehensively an outline to obtained a discrete-time state space model for linear system on a modified quadruple tank system in a simple and constructive method. This lab scale system represents a MIMO system which has complicated variables interactions and complex control problems. The dynamics of the system is described by an existing simulation models in terms of deterministic and stochastic non-linear continuous time models. These models were linearized and discretized in order to form a discrete-time linear time-invariant difference equations, the form that is used in the Kalman Filter for estimations. Based on the model and measurements, the current state of the system was estimated and in additional, the comparison between dynamic and static Kalman filter was also presented.

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# PAPER C

# Model based control implementation of a modified quadruple tank system using a predictive control strategy

**Authors:**

Sazuan Nazrah Mohd. Azam and John Bagterp Jørgensen

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# Model Based Control Implementation of a Modified Quadruple Tank System Using a Predictive Control Strategy

Sazuan Nazrah Mohd. Azam and John Bagterp Jørgensen

*Abstract*—**The implementation of a model based controller for a modified quadruple tank system (MQTS) is addressed in this paper. The purpose is to demonstrate the application of Model Predictive Controller (MPC) to a multi input multi output (MIMO) system that has complicated variables interactions. An existing dynamics of the system that has deterministic and stochastic components described as a linear discrete-time state space model is employed for the purpose of study. The MPC consists of a state estimator and a constrained regulator. A Kalman filter is incorporated to estimate the current state from the filtered part and the predictions part is used by constrained regulator, an optimal control problem (OCP) to predict the future output trajectory. The objective of the OCP consists of a tracking error term that penalizes deviations of the predicted outputs from the setpoint and a regularization term that penalizes the changes in the inputs (manipulated variables). The resulting OCP is represented as a QP is solved and the performance of MPC is demonstrated through simulations using MATLAB.**

## I. Introduction

Model based controller is one of the advanced control strategy that is currently common and extensively recognized in industry and academic, famously known as Model Predictive Control (MPC). The MPC first breakthrough is from a seminal publication of Model Predictive Heuristic Control [1] and later [2] who came out with Dynamic Matrix Control.

MPC is a controller that utilizes the identified model of a system to predict its future behaviour, given a prediction horizon. The main idea is to minimize the cost function and taking into account the constraints. Then the first controller moves is implemented at a sampling instants over the control horizon, by implementing only the first move the optimal feedback is achieved and then the complete sequence will be repeated again, which is known as moving horizon concept [3]. This flexibility is one of the important significant advantages of MPC [4]. Nowadays the application of MPC are not limited to the process control field, but also including other various fields.

In this paper, we focus on implementing the predictive control strategy on a lab scale system that exhibits the characteristic of a complex MIMO system. The modified quadruple tank system (MQTS) which is inspired by [5] has been widely used for education in demonstrating advanced control strategies [6], [7] and will be use through out this work as an example to assimilate the fundamental theory of MPC.

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S. N. Mohd. Azam and J. B. Jørgensen are from Department of Applied Mathematics and Computer Science, Technical University of Denmark, DK-2800 Kgs. Lyngby, Denmark, {snaz, jbjo}@dtu.dk

Several strategies of controllers is implemented on the quadruple tank system such as [8] and [9], while [10] and [11] has been extensively described the application of MPC on the quadruple tank system with different approaches. A comparative study of model based control for the four-tank system using IMC and DMC is provided by [9] and a year later an analysis of robust control for the identical system is done [12]. The Kalman filter is incorporated for state estimation to obtained an optimal MPC as shown in [13] and specified in detail in [14] attained from the Discrete-time Algebraic Riccati Equation.

One of MPCs advantages is the ability to work within certain constraints, therefore the main purpose of this work is to demonstrate the implementation of unconstrained and input constrained MPCs complete with the derivation of the equations. To solve the OCP, we express the control task which is tracking of the setpoint trajectory as a quadratic optimization problem by performing several experiments and simulations studies for observation to demonstrate the versatility of the advanced controller.

The outline of this paper is as follows. A brief introduction and description of the MQTS is presented in Section II. Next, the details of the control structure of the process and the implementation of unconstrained and input constrained MPC is shown in detail in Section III followed by the results and discussion of the closed loop simulations in Section IV. Lastly, this paper is concluded in the final section V.

## II. The Modified Quadruple Tank System

The MQTS as shown in Fig. 1 consist of four identical tanks and two pumping systems. It is a simple process that is non-linear MIMO but demonstrates a complicated interactions between these variables including the manipulated and controlled variables. Flows through the pump $F_1$ and $F_2$ can be controlled in order to achieve desired setpoints of $r_1$ and $r_2$ of water levels in Tank 1 and Tank 2, respectively. Whereas $F_3$ and $F_4$ are stochastic variables and assumed to be normally distributed, hence cannot be controlled. The target is to control the level of the water in Tank 1 and Tank 2 by adjusting the flow rates $F_1$ and $F_2$ which are distributed across all four tanks. The height of the water level in these two tanks, $h_1$ and $h_2$ is measured and controlled.

The variables is defined as $x$ indicates the state variables, $y$ is the measured variables, $u$ indicates the manipulated variables (MVs), $z$ is the controlled variables (CVs) and $d$

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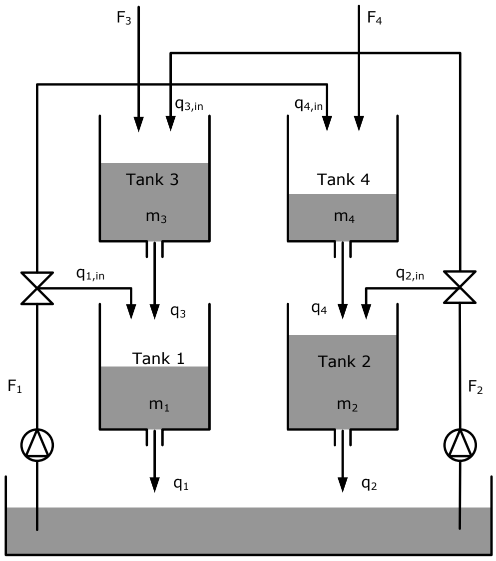


Fig. 1: Schematic diagram of the modified quadruple tank process

is the disturbances. The dynamic of the process is described in Stochastic Nonlinear Model (SDE) given as:

$$d\mathbf{x}(t) = f(\mathbf{x}(t), u(t), d(t), p)dt + \sigma d\mathbf{w}(t) \tag{1a}$$

$$\mathbf{y}(t) = c(\mathbf{x}(t)) + \mathbf{v}(t) \tag{1b}$$

$$\mathbf{z}(t) = c(\mathbf{x}(t)) \tag{1c}$$

subject to $w_t \sim N(0, R_w)$ and $v_t \sim N(0, Rv)$. The modeling part of the MQTS is fully described in [15]. Considering the stochastic part of the model, a piecewise constant process noise $w$ , measurement noise $v$ and uncertainty of the initial state $x_0$ to the process is added. The stochastic nonlinear model from equation (1) is realized and expanded into a linear discrete time state space model as in the equation below

$$x_{k+1} = A_d x_k + B_d u_k + E_d(d_k + w_k) \tag{2a}$$

$$y_k = C_d x_k + v_k \tag{2b}$$

$$z_k = C_{dz} x_k + v_k \tag{2c}$$

subject to $x_0 \sim N(\bar{x_0}, P_p)$, $w_k \sim N(0, Q)$ and $v_k \sim N(0, R)$. In order to rewrite the difference equation system representation (2) with disturbances in a more structured form, the Markov parameters is introduced. and the state space model can be represented in a matrix form of

$$Y = \Phi x_0 + \Gamma_u U + \Gamma_d D \tag{3}$$

For full description of the discrete linear state space model realization and Markov parameters description, refer [16]. From equation (3), $\Phi$ and $\Gamma$ will be used for the prediction part from the Kalman filter for the MPC.

## III. Model Predictive Control

Referring to the block diagram of the control structure for the MQTS in Fig. 2, the model predictive control part consists of an estimator which estimates the current states of the system, $\hat{x}$ given the previous measurement from the process, $y$ and a regulator which minimizes the difference between the reference values $r_1, r_2$ and the controlled variables $z_1, z_2$ with respect to the manipulated variables $u_1, u_2$. The output of the regulator which is the new input variables is then fed to the MQTS thus yielding a new state and a new measurement. This process is iteratively repeated for a specified timespan or until a certain stopping criteria is reached in a closed loop manner. From the diagram it is possible to visualise the implementation of the MPC that will yield the optimum input to the system which would result in an approach to the desired reference values.

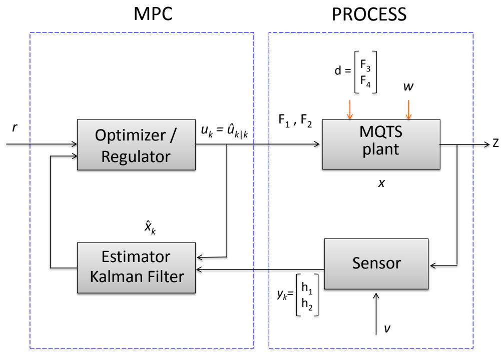


Fig. 2: Block diagram of the control structure for the MQTS process

### A. State Estimation for the Discrete-Time Linear System

To obtained the current state estimation and the future output predictions, a static Kalman filter is utilised throughout this work due to the fact that it will lighten the computations of the controller since the covariance matrix $P_{k|k-1}$ is kept constant [16]. From [16] the linear discrete-time stochastic model is given as

$$x_{k+1} = A_d x_k + B_d u_k + E_d(d_k + w_k) \tag{4a}$$

$$y_k = C_d x_k + v_k \tag{4b}$$

subject to where $x_0$ is realised as a stochastic variable, in order to achieve a full stochastic simulation. It is taken that $x_0 \sim N(0, P)$. The other stochastic, namely $w_k$ and $v_k$, are realised considering a possible correlation between them where the process noise $w_k$ and measurement noise $v_k$ are distributed as

$$\begin{bmatrix} w_k \\ v_k \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} Q_k & S_k \\ S^T{}_k & R_k \end{bmatrix}\right) \tag{5}$$

Here, $Q_k$ and $R_k$ are the variances of the corresponding stochastic variables $w_k, v_k$ and $S_k$ denotes the correlation between these variables which for the purpose of this study is taken as 0.

By the assumption of the covariance matrix $P_{k|k-1}$ is constant, it also allows the computation to be carried out as Ricatti equation. The stationary one-step ahead state error covariance matrix obtained from the Discrete-time Algebraic Riccati Equation (DARE) is given by

$$\begin{aligned} P = APA^T + Q - (APC^T + S)(CPC^T + R)^{-1} \\ (APC^T + S)^T \end{aligned} \tag{6}$$

Assuming a stationary approach allows to calculate part of the algorithm off-line,

$$R_e = CPC^T + R \tag{7a}$$
$$K_{fx} = PC^T R_e^{-1} \tag{7b}$$
$$K_{fw} = SR_e^{-1} \tag{7c}$$

and the remaining part of the algorithm, namely measurement and prediction, is computed for each time step

$$e_k = y_k - \hat{y}_{k|k-1} \tag{8a}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx,k} e_k \tag{8b}$$
$$\hat{w}_{k|k} = K_{fw} e_k \tag{8c}$$
$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k + \hat{w}_{k|k} \tag{8d}$$

### *B. Unconstrained MPC*

In this section we develop unconstrained model predictive controllers based on the discrete time state space models as in equation (4). Having designed the Kalman filter for the MQTS, the regulator will be implemented to form the complete MPC framework in Fig. 2.

The control task is to track the setpoint trajectory as a quadratic optimization problem by developing an objective function that will minimize the deviation of predicted output trajectory from the setpoint trajectory,

$$\min_u \quad \frac{1}{2}\sum_{k=1}^{N} ||z_k - r_k||^2_{Q_z} + \frac{1}{2}\sum_{k=0}^{N-1} ||\Delta u_k||^2_S \tag{9a}$$

s.t.

$$x_{k+1} = Ax_k + Bu_k + E_{d_k} \quad k = 0, 1, \ldots, N-1 \tag{9b}$$
$$z_k = C_x x_k \quad k = 0, 1, \ldots, N-1 \tag{9c}$$

From equation (3), $Z$ can be expressed in a matrix form of

$$Z_k = \Phi x_0 + \Gamma_u U + \Gamma_d D \tag{10}$$

where $Z_k$ and $R_k$ are

$$Z_k = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} \quad R_k = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix}$$

$U_k$, $R_k$ and $Z_k$ are deviation variables vectors. The weight matrices $Q_z$ and $S$ are realised as diagonal matrices since we want to penalise the deviation of Tank 1 and 2 from the desired targets, as well as large steps in the input variables, respectively.

In minimizing the objective function equation (9) it can be expressed in a compact form as

$$min \quad \phi = \phi_z + \phi_{\Delta u} \tag{11}$$

The first term in the objective function is related to the desired target and the main part of the least squares minimization problem. It ensures that the system reaches towards the desired target values $r_1, r_2$.

$$\begin{aligned} \phi_z &= \frac{1}{2}||Z_k - R_k||^2_{Q_z} \\ &= \frac{1}{2}\left(\Phi x_0 + \Gamma U_k + \Gamma_d D - R_k\right)^2 \\ &\quad Q_z\left(\Phi x_0 + \Gamma U_k + \Gamma_d D - R_k\right) \end{aligned} \tag{12}$$

Let

$$b_k = R_k - \Phi x_0 - \Gamma_d D \tag{13}$$

and now we can expressed the problem as QP in minimizing $\phi_z$, an objective function based on the controlled variables.

$$\begin{aligned} \phi_z &= \tfrac{1}{2}\left(\Gamma U_k - b_k\right)^T Q_z \left(\Gamma U_k - b_k\right) \\ &= \tfrac{1}{2}U_k^T \Gamma^T Q_z \Gamma U_k - (\Gamma^T Q_z b_k)^T U_k + \rho \\ &= \tfrac{1}{2}U_k^T H_z U_k + g_z^T U_k + \rho \end{aligned} \tag{14}$$

where

$$H_z = \Gamma^T Q_z \Gamma \tag{15a}$$
$$\begin{aligned} g_z &= -\Gamma^T Q_z b_k \\ &= -\Gamma^T Q_z R_k + \Gamma^T Q_z \Phi_{x_0} + \Gamma^T Q_z \Gamma_d D \\ &= M_R R_k + M_{x_0} x_0 + M_d D \end{aligned} \tag{15b}$$
$$\rho = \frac{1}{2}\left(b^T Q_z b_k\right) \tag{15c}$$

Since using the objective function based only on the controlled variables is insufficient, we include the input variables, $\phi_{\Delta u}$ in the objective function as the second term. The second term is the regularization term, which ensures smooth input solutions which minimizes the difference of $u_k$ from the previous input so that to have less error.

$$min \quad \phi_{\Delta u} = \frac{1}{2}\sum_{k=0}^{N-1} ||\Delta u_k||^2_S \tag{16}$$

We want to rewrite this problem into standard QP. First we want to derive $\phi_{\Delta u}$ using similar approach to $\phi_z$ but $\Delta u$ needs to be expressed in terms of $u_k$,

$$\Delta u_k = \begin{bmatrix} \Delta u_0 \\ \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 - u_0 \\ u_2 - u_1 \\ u_3 - u_2 \end{bmatrix} - \begin{bmatrix} u_{-1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

To have better formulation we introduce $\Lambda$, $U_k$ and $I_0$ as

$$\Lambda = \begin{bmatrix} I & 0 & 0 & 0 \\ -I & I & 0 & 0 \\ 0 & -I & I & 0 \\ 0 & 0 & -I & I \end{bmatrix} \quad U_k = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad I_0 = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $I$ denotes an identity matrix of the size of $u$, $I_0$ denotes the block vector with $I$ in the first entry while having zero matrices fill up the rest of the rows so that to have the same row-size as matrix $\Lambda$, then $\Delta u_k$ can be rewritten as

$$\Delta u_k = \Lambda U_k - I_0 u_{-1} \tag{18}$$

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We expand equation (16) by substituting with equation (18) and the objective function $\phi_{\Delta u}$ yields

$$\begin{aligned} \phi_{\Delta u} &= \tfrac{1}{2}||\Lambda U_k - I_0 u_{-1}||_S^2 \\ &= \tfrac{1}{2}U_k^T \left(\Lambda^T \bar{S}\Lambda\right) U_k + \left(-\Lambda^T \bar{S} I_0 u_{-1}\right)^T U_k + \rho \\ &= \tfrac{1}{2}U_k^T H_{\Delta_u} U_k + g_{\Delta_u}^T U_k + \rho \end{aligned} \tag{19}$$

where

$$H_{\Delta_u} = \Lambda^T \bar{S}\Lambda \tag{20a}$$

$$\begin{aligned} g_{\Delta_u} &= -\Lambda^T \bar{S} I_0 u_{-1} \\ &= M_{u-1}u_{-1} \end{aligned} \tag{20b}$$

$$\rho = \frac{1}{2}\left(U_k^T I_0^T \bar{S}\right) \tag{20c}$$

Combining equation (14) and (19) and $\rho$ is disregarded as these are constant, resulting an optimization of the deviation from the setpoint (reference trajectory) and the input variables given as equation (9) becomes

$$\min_u \quad \frac{1}{2}U_k^T H U_k + g^T U_k \tag{21a}$$

s.t.

$$x_{k+1} = Ax_k + Bu_k + E_{d_k}, \quad k = 0, 1, \ldots, N-1 \tag{21b}$$

$$z_k = C_x x_k, \quad k = 0, 1, \ldots, N \tag{21c}$$

where

$$H = H_z + H_{\Delta_u} \qquad g = g_z + g_{\Delta_u} \tag{22}$$

MPC computation when stated as a QP is solved by first computing

$$g = M_R R_k + M_{x_0}x_0 + M_d D + M_{u-1}u_{-1} \tag{23}$$

then solving the QP

$$\begin{aligned} u^* &= -H^{-1}g \\ &= -H^{-1}\left(M_R R_k + M_{x_0}x_0 + M_d D + M_{u-1}u_{-1}\right) \\ &= L_R R + L_{x_0}x_0 + L_d D + L_{u-1}u{-1} \end{aligned} \tag{24}$$

with the first block row of $L_R, L_{x_0}, L_d, L_{u-1}$ is given by $K_R, K_{x_0}, K_d, K_{u-1}$ and the optimal control law is

$$u_0^* = K_R R + K_{x_0}x_0 + K_d D + K_{u-1}u{-1} \tag{25}$$

### C. Input Constrained MPC

To make the simulation of the model based controller on the MQTS more realistic, we considered two different hard constraints; first is the one that sets the upper and lower bounds on the manipulated variables, $u_{min} \leqslant u \leqslant u_{max}$ and the other one is the rate of change in input, $\Delta u_{min} \leqslant \Delta u \leqslant \Delta u_{max}$. The formulation of the problem is the same as stated in equation (9) but subjected to constraint it becomes

$$\min_u \quad \frac{1}{2}\sum_{k=1}^{N}||z_k - r_k||_{Q_z}^2 + \frac{1}{2}\sum_{k=0}^{N-1}||\Delta u_k||_S^2 \tag{26a}$$

s.t.

$$x_{k+1} = Ax_k + Bu_k + E_{d_k} \quad k = 0, 1, \ldots, N-1 \tag{26b}$$

$$z_k = C_x x_k \quad k = 0, 1, \ldots, N-1 \tag{26c}$$

$$u_{min} \leqslant u_k \leqslant u_{max} \quad k = 0, 1, \ldots, N-1 \tag{26d}$$

$$\Delta u_{min} \leqslant \Delta u_k \leqslant \Delta u_{max} \quad k = 0, 1, \ldots, N-1 \tag{26e}$$

In standard QP form, this can be written as (21) and the inequality constraints referring to equation (26e) needs to be updated at each iteration since it contains the input variables form the previous step, $u_{k-1|k}$. In developing the simulation code for the input constraint MPC, the Hessian matrix is obtained by offline computations, essentially the same as the unconstrained MPC but the calculation of the inequality matrix and the corresponding upper and lower bounds are set beforehand. Likewise, during the regulator process, equation (26d) is supplied as upper and lower bounds to the quadratic solver in Matlab and the 2 inequalities in equation (26e) are formulated in the form of

$$\begin{bmatrix} \Delta u_{min} \\ \Delta u_{min} \\ \vdots \\ \Delta u_{min} \end{bmatrix} \leqslant \begin{bmatrix} u_0 - u_{-1} \\ u_1 - u_0 \\ \vdots \\ u_N - u_{N-1} \end{bmatrix} \leqslant \begin{bmatrix} \Delta u_{max} \\ \Delta u_{max} \\ \vdots \\ \Delta u_{max} \end{bmatrix} \tag{27}$$

Since the first row contains $u_{-1}$ this can be written as

$$\Delta u_{min} + u_{-1} \leqslant u_0 \leqslant \Delta u_{max} + u_{-1} \tag{28a}$$

$$u_{min} \leqslant u_0 \leqslant u_{max} \tag{28b}$$

and the rest of the rows are arranged in the form of

$$\begin{bmatrix} \Delta u_{min} \\ \Delta u_{min} \\ \vdots \\ \Delta u_{min} \end{bmatrix} \leqslant \begin{bmatrix} -I & I & & & \\ & -I & I & & \\ & & \ddots & \ddots & \\ & & & -I & I \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \leqslant \begin{bmatrix} \Delta u_{max} \\ \Delta u_{max} \\ \vdots \\ \Delta u_{max} \end{bmatrix} \tag{29}$$

which can be simplified as

$$\Delta U_{min} \leqslant \Lambda U_k \leqslant \Delta U_{max}$$

Therefore, an optimization of the deviation from the setpoint (reference trajectory) and the input variables becomes

$$\min_u \quad \Phi = \frac{1}{2}U_k^T H U_k + g^T U_k \tag{30a}$$

s.t.

$$U_{min} \leqslant U_k \leqslant U_{max} \tag{30b}$$

$$\Delta U_{min} \leqslant \Lambda U_k \leqslant \Delta U_{max} \tag{30c}$$

where $H$ and $g$ is as given in (22). The optimised input is returned and the first two entries obtained and applied in the next iteration, similar to the previous unconstrained MPC.

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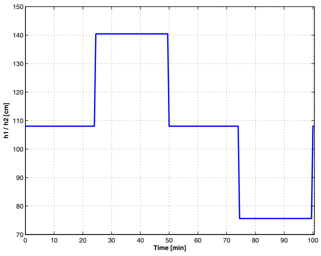


Fig. 3: Reference Trajectory

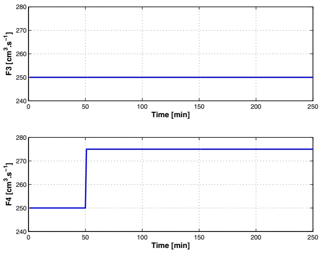


Fig. 4: $F_4$ Step Input

## IV. Closed Loop Simulation

In this section the MPCs presented in previous section are discussed and evaluated with various test cases. The main objective is to evaluate the performance of the MPC in terms of the behaviour of the system and to verify should the realisations are physically feasible. For the simulation of the MQTS, the linear stochastic model is utilised in which noise is included as a normal distribution to the disturbance variables and the measurement variables. For all MPC implementations, the reference trajectory is given as a multi step input changes as in Fig. 3 where each step input interval allows the system to reach at certain steady state before yielding the next step change. The disturbance variables are initialised such that they contain a step input change as in Fig. 4 in order to evaluate the performance of the regulator in compensating for the upcoming disturbance. Results and analysis of the simulations for each MPCs which was run for 200 minutes of simulation time is discussed in the sections below.

### A. Unconstrained MPC

The unconstrained MPC is simulated on the linearised model of MQTS with the above described conditions, influenced by white process noise and measurement noise. Fig. 5 and Fig.6 depicts the output and control variable for both multi step input changes of $h_1$ and $h_2$ respectively. The responses shows that the system is able to track the references with minimal overshoot and small transient deviations, keeping the desired height levels of Tank 1 and Tank 2 at the desired setpoints. Although the controller managed to compensate the disturbances, the flow rate in $F_1$ elongated below 0 $cm^3.s^{-1}$ which indicates the suction of pump. This glitch can be solved by implementing the next MPC strategy.

### B. Input Constrained MPC

A similar closed loop simulation was implemented but in this section, the constraints for the input is included. Fig. 7 represents the output of the simulation in response to a multiple setpoint changes in $h_1$ with the same condition of reference trajectory and disturbances as described earlier and Figure 8 represents the output of the simulation in response to a multiple setpoint changes in $h_2$. The considered constraints is given as

$$\begin{array}{rcccc} 0 & \leqslant & u_{min} & \leqslant 450 & (cm^3/s) \\ -20 & \leqslant & \Delta u_k & \leqslant 20 & (cm^3/s) \end{array}$$

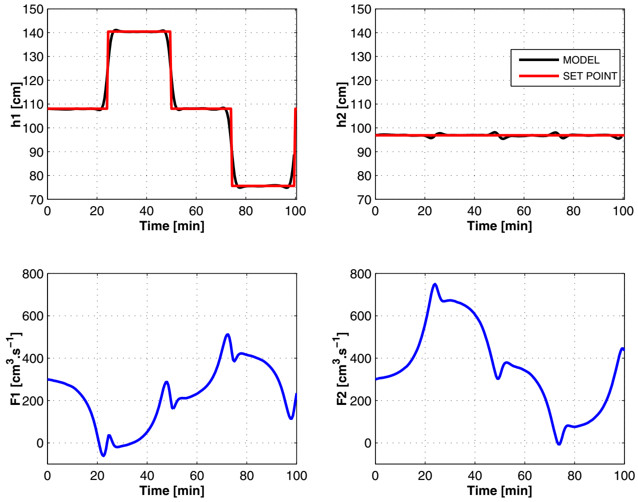


Fig. 5: Output of the simulation in response to a multiple setpoint changes in $h_1$ with a setpoint change in $d_2$ for Unconstrained MPC

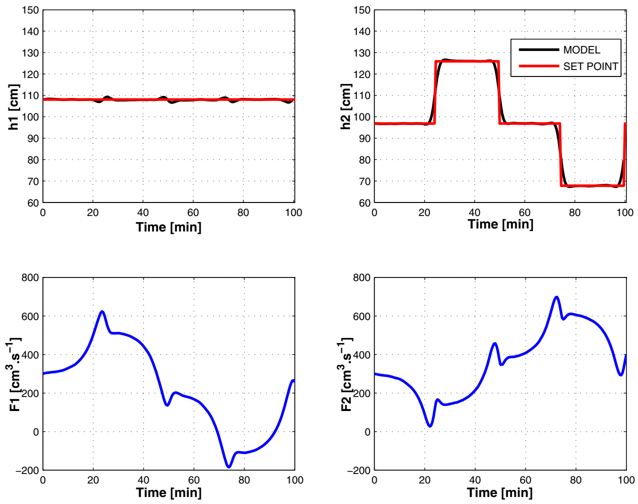


Fig. 6: Output of the simulation in response to a multiple setpoint changes in $h_2$ with a setpoint change in $d_2$ for Unconstrained MPC

From both of the responses it can be seen that the transient properties is slightly deteriorated but with acceptable performance degradation although the setpoint tracking is unable to reach at certain points due to compensation from the input constraints and the capacity of the plant. Referring to the flow of $F_1$ and $F_2$ for both step changes of $h_1$ and $h_2$ simulations, it is noticeable that the water flow is within the boundaries, indicating that the rate constraints results in a more well behaved flow characteristics with a slight loss of transient properties and reference tracking. With input constraints, the controller is capable of operating within the limit bounds but with an acceptable drawback of reaching the target setpoint due to the capacity of the pumps and the amount of computation required is higher.

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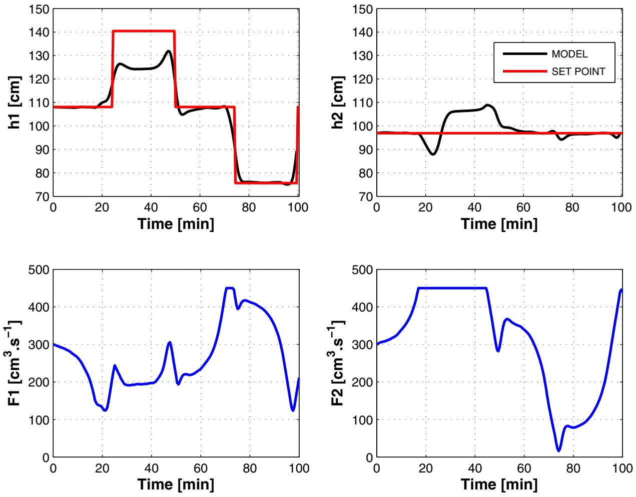


Fig. 7: Output of the simulation in response to a multiple setpoint changes in $h_1$ with a setpoint change in $d_2$ for Input Constrained MPC

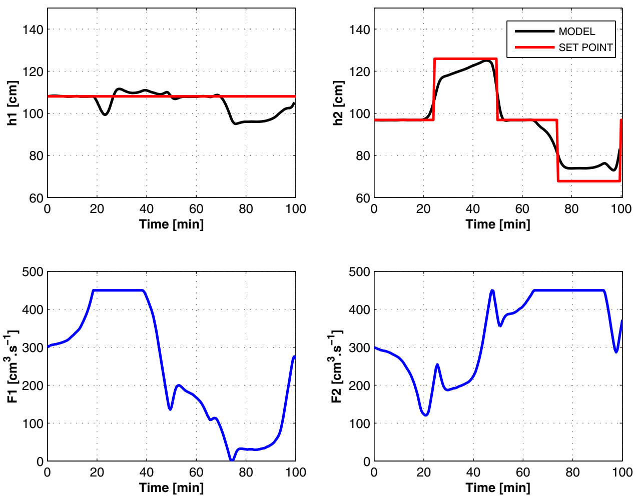


Fig. 8: Output of the simulation in response to a multiple setpoint changes in $h_2$ and a setpoint change in $d_2$ for Input Constrained MPC

## V. Conclusion

This paper has described comprehensively an outline for MPC implementation for linear system on a MQTS in a simple and constructive method. This lab scale system represents a MIMO system which has complicated variables interactions and complex control problems. The dynamics of the system is described by an existing simulation models in terms of a discrete-time linear time-invariant difference equations, the form that is used in the Kalman Filter for estimations and predictions. Based on the model and measurements, the current state of the system was estimated and the predictions part was used to predict the future output trajectory. Given an input trajectory, the constraint regulator used the predictions part from the Kalman Filter for error tracking for set point changes and as a regulator to compensate input changes. The QP which was from the OCP was solved and the performance of the MPC was demonstrated through simulations. The implementation of unconstrained and input constrained MPC for the MQTS is compared.

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